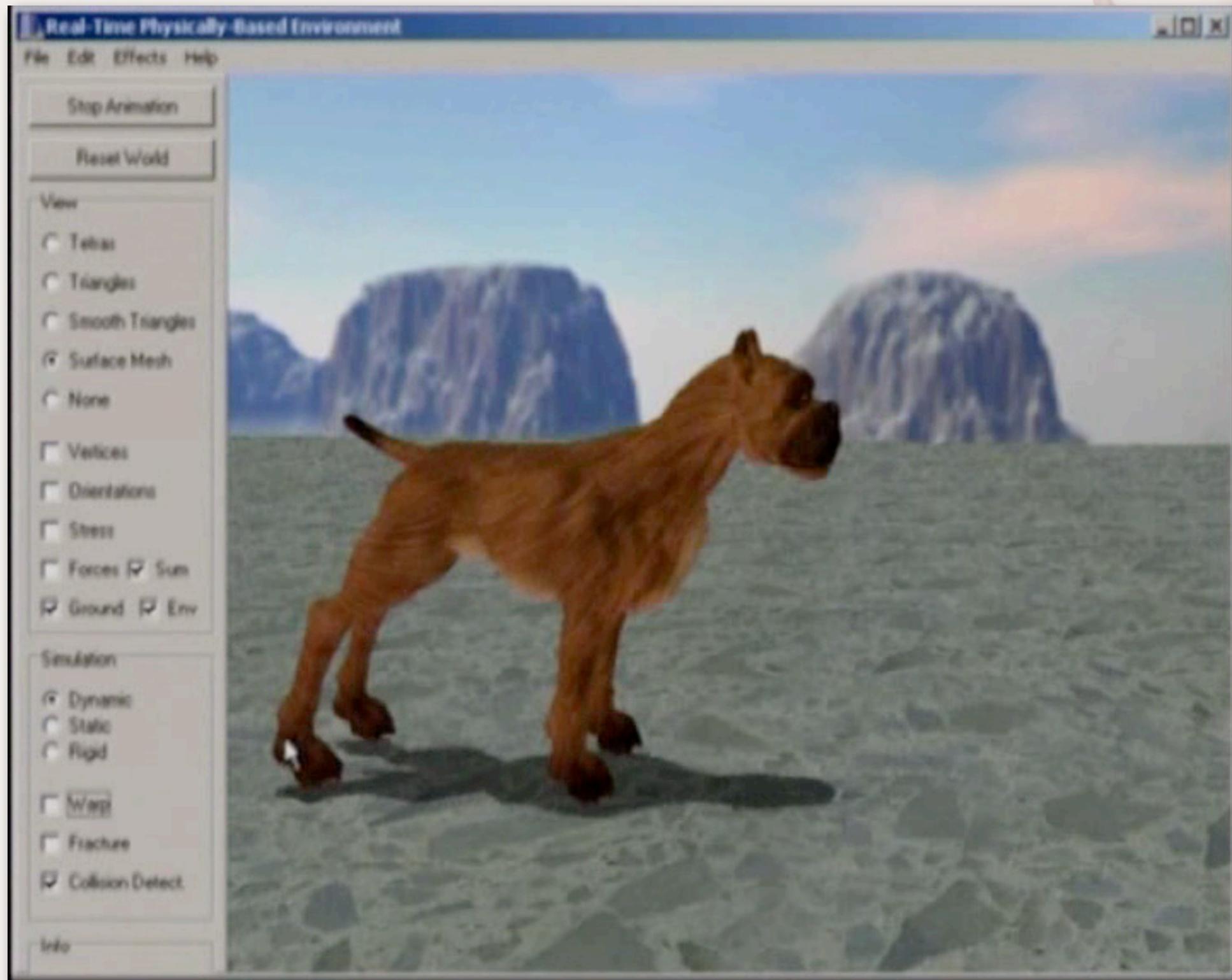


Deformable Materials 2

Adrien Treuille

source: Müller, Stam, James, Thürey. *Real-Time Physics Class Notes*.

Goal



Overview

- **Strain (Recap)**
- **Stress**
- **From Strain to Stress**
- **Discretization**
- **Simulation**



Overview

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Deformations

Spring deformed by Δx :



$$\text{force} = -E \Delta x$$

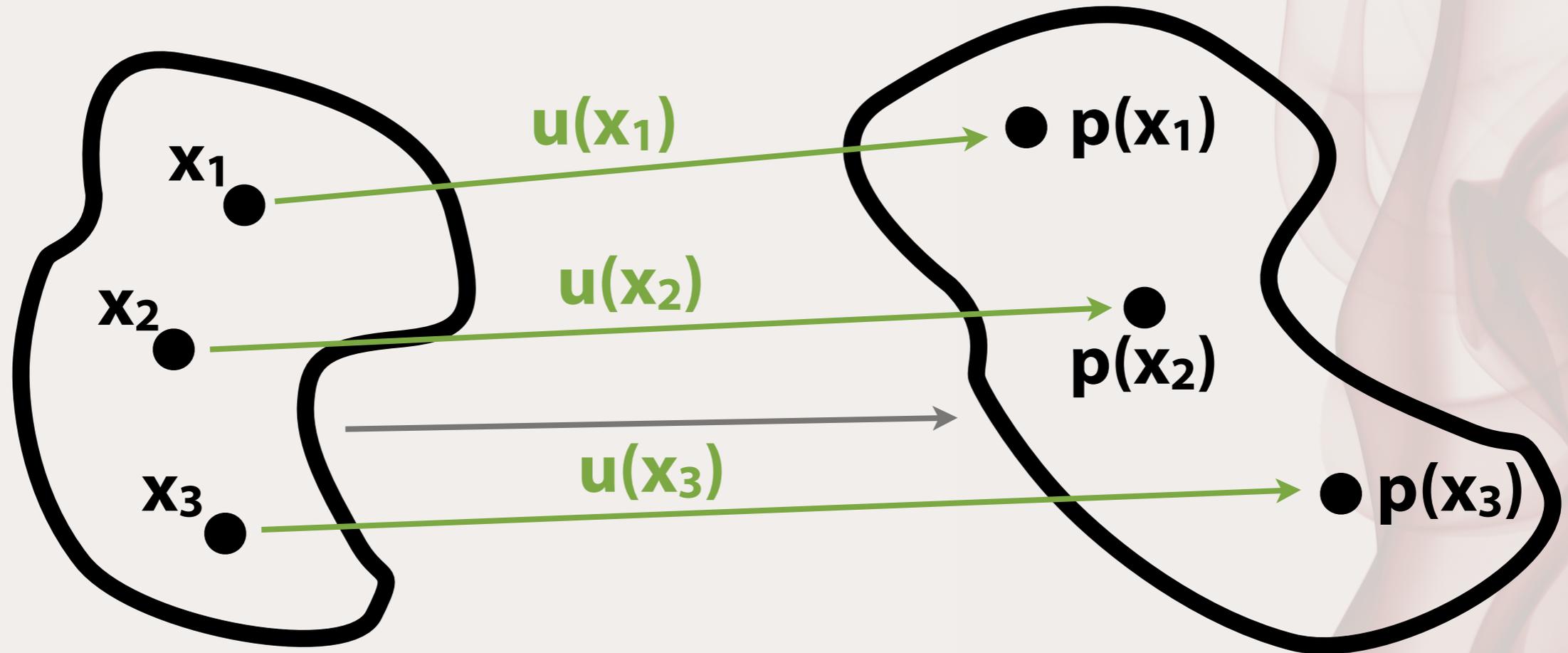
stress: σ Young's modulus strain: ϵ

Hooke's Law:

$$\sigma = E \epsilon$$



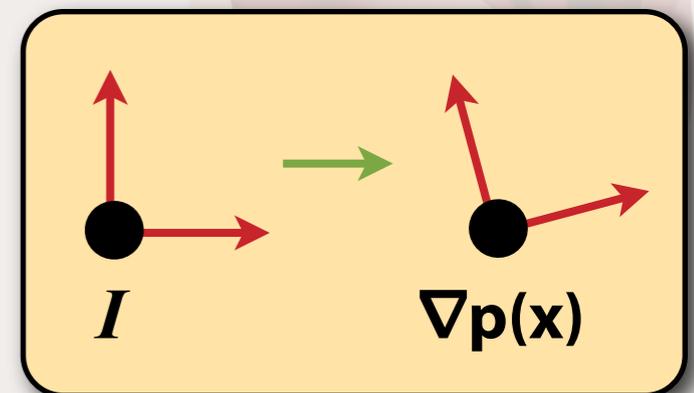
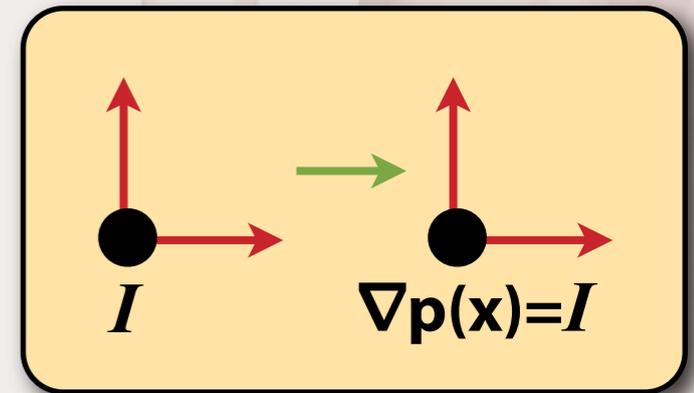
In 3D...



$$u(x) = p(x) - x$$

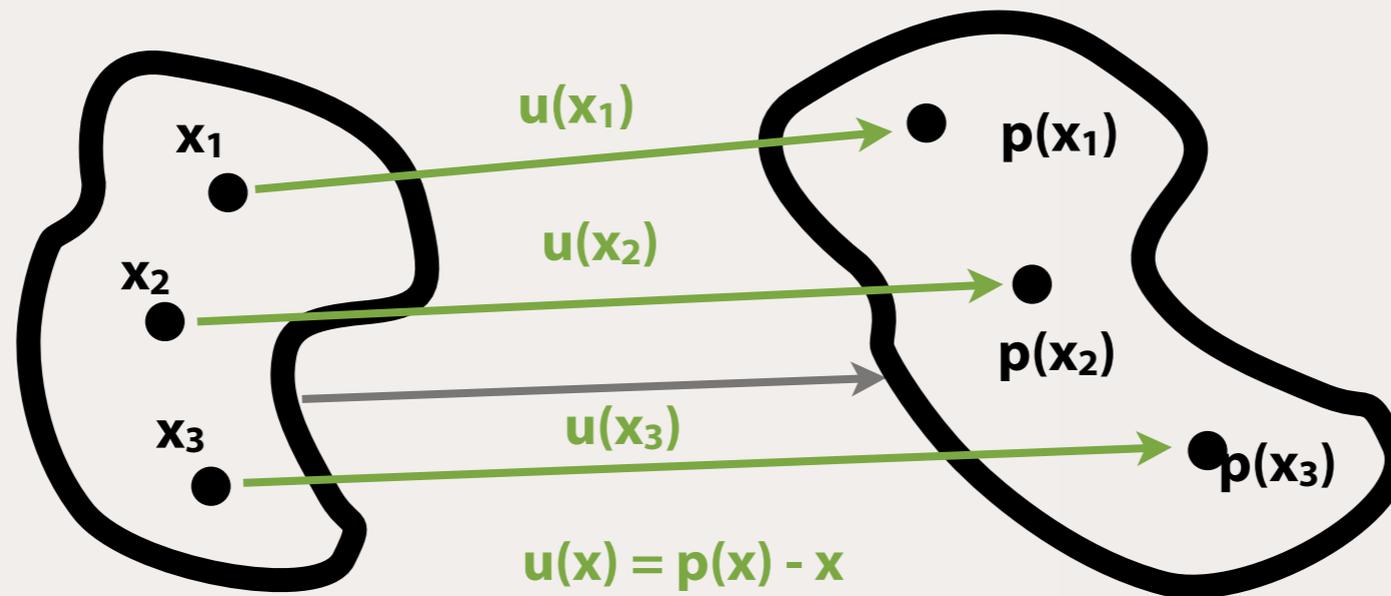
Defining Strain

- Strain is **invariant to translation**.
 - Ignore $p(x)$
 - Define in terms of local coordinate system transform: $\nabla p(x)$.
- Strain is **invariant to rotation**.
 - If $[\nabla p(x)]^T \nabla p(x) = I$,
 - Then $\epsilon = 0$
- Natural to define strain as:
 - $\epsilon = \frac{1}{2}([\nabla p(x)]^T \nabla p(x) - I)$
 - 6 DOFs



$$\epsilon = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix}$$

Green's Strain



$$\epsilon_G = \frac{1}{2} \left(\nabla u + [\nabla u]^T + [\nabla u]^T \nabla u \right)$$

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Stress

$$\sigma = E \epsilon$$



Stress

Direct Stress:

Direct stresses cause compression.

$$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$$

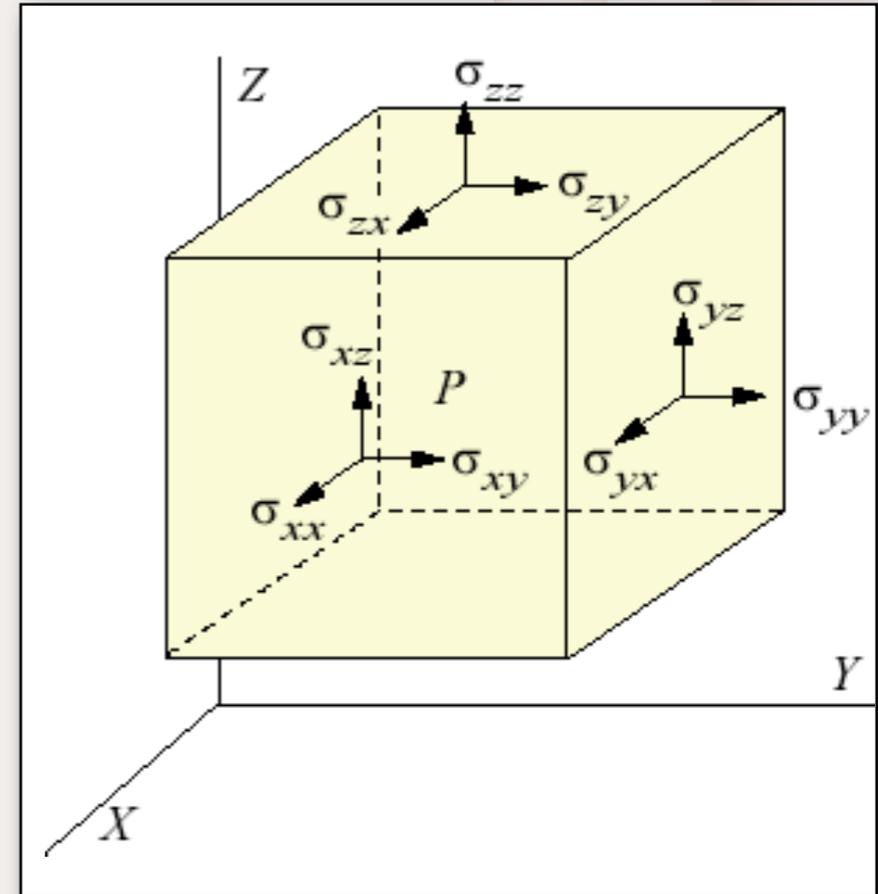
Shear Stress:

Shear stresses resist compression.

$$\sigma_{xy}, \sigma_{yz}, \sigma_{xz}$$

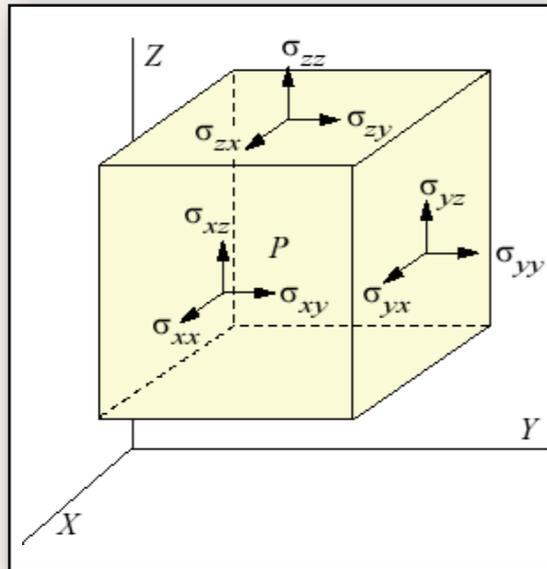
Stress Tensor:

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$$



source: http://www.efunda.com/formulae/solid_mechanics/mat_mechanics/stress.cfm

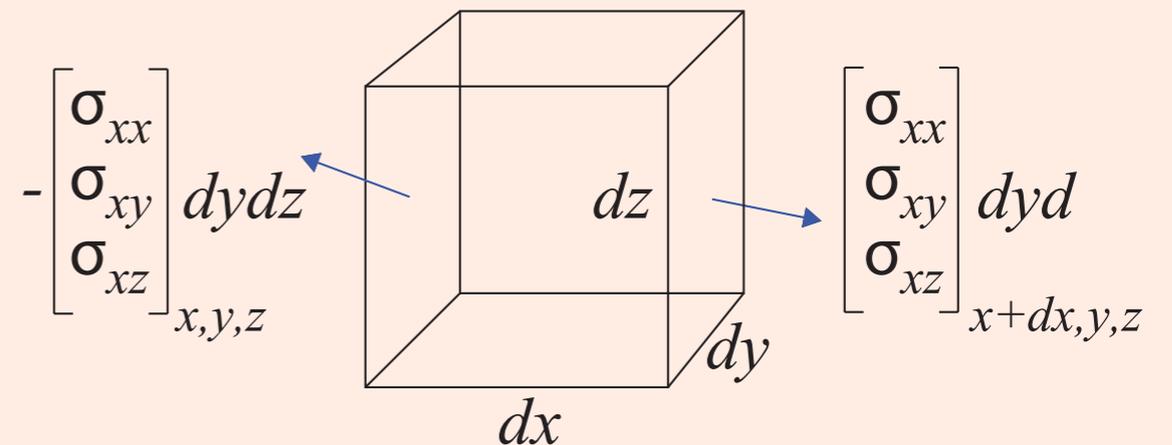
Stress Tensor Interpretation



$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$$

Stress measures the **force on each face**:

$$\frac{d\mathbf{f}}{dA} = \boldsymbol{\sigma} \cdot \mathbf{n}.$$



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Young's Modulus

$$\sigma = E \epsilon$$



Voigt Notation

Stress

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$$

$$\{\sigma\} = [\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{xz}]^T \in \mathbf{R}^6$$

Strain

$$\epsilon = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix}$$

$$\{\epsilon\} = [\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{xy}, \epsilon_{yz}, \epsilon_{xz}]^T \in \mathbf{R}^6$$

Isotropic Materials

$$\{\sigma\} = E\{\epsilon\} \quad E \in \mathbf{R}^{6 \times 6}$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{xy} \\ \epsilon_{yz} \\ \epsilon_{zx} \end{bmatrix}$$

E

Elastic Stiffness

How strongly the material resists deformation.

$\nu \in \left[0, \frac{1}{2}\right)$

Poisson's Ratio

How much volume is conserved.

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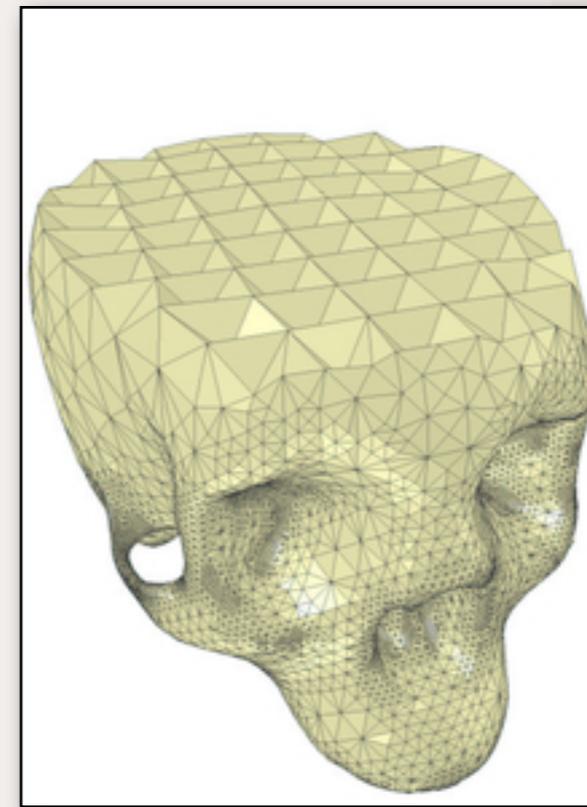
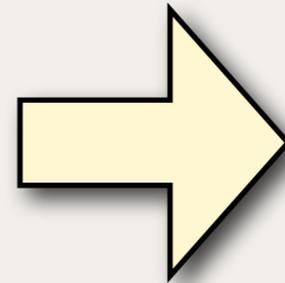
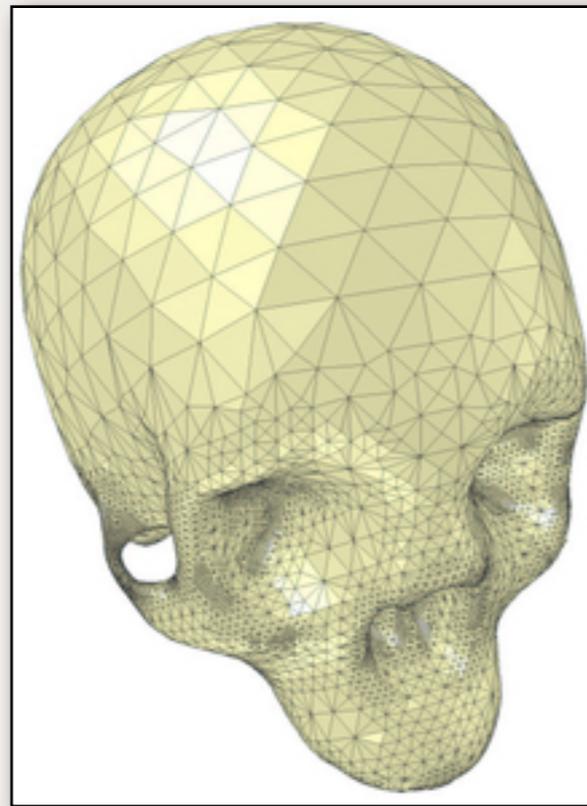
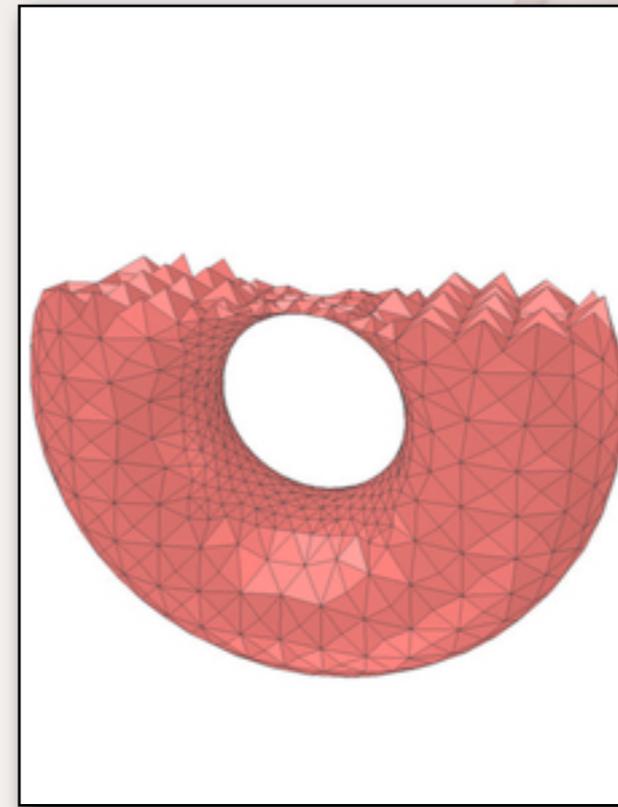
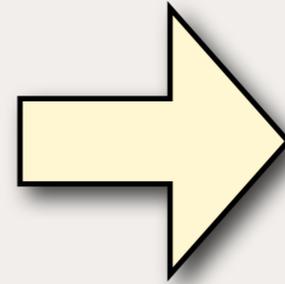
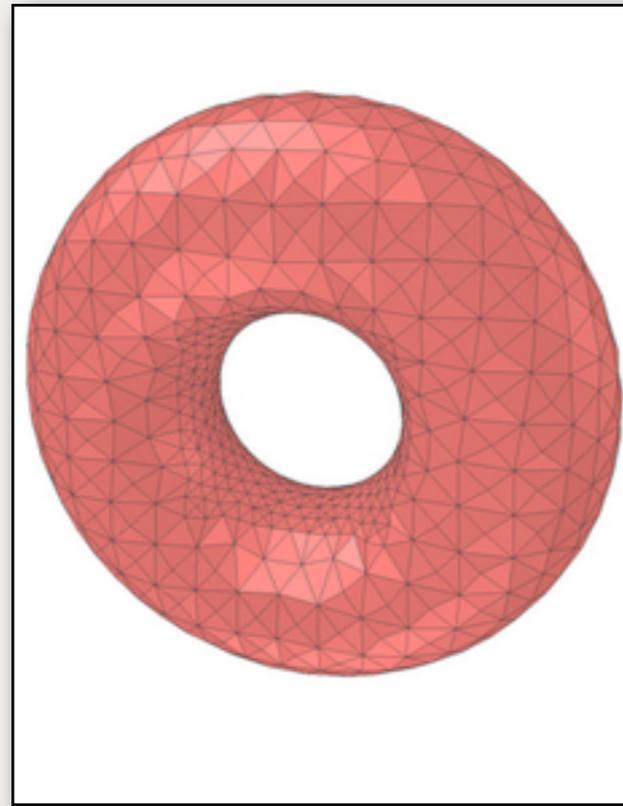


Overview

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- **Discretization**
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Discretization



source: Bridson, R., Teran, J., Molino, N. and Fedkiw, R., "Adaptive Physics Based Tetrahedral Mesh Generation Using Level Sets", *Engineering with Computers* 21, 2-18 (2005).

Overview

- Strain (Recap)
- Stress
- From Strain to Stress
- **Discretization**
- Simulation

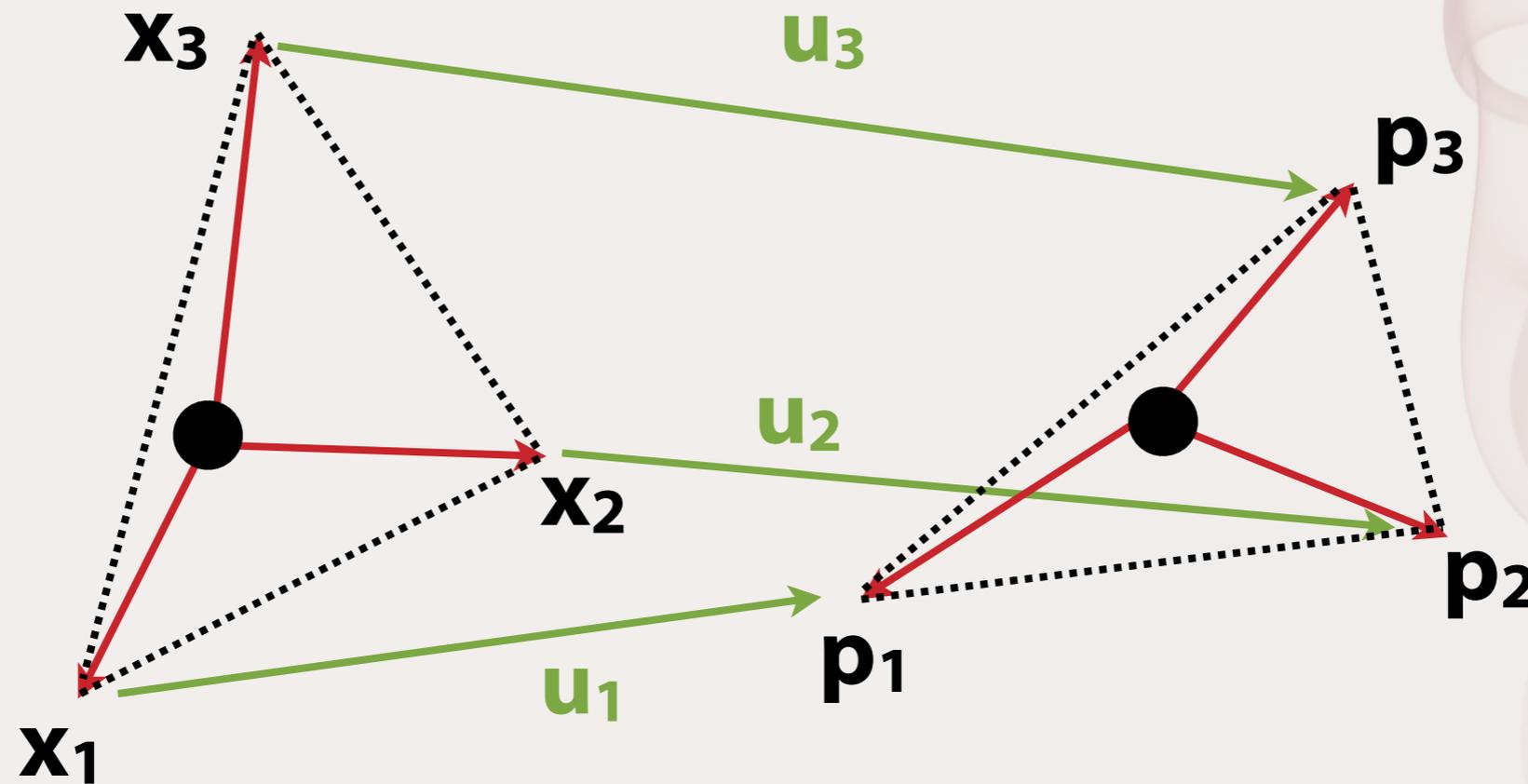


Overview

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- **Simulation**



Discrete Strain



$$\mathbf{p}(\mathbf{x}) = [\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3] [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]^{-1} \mathbf{x} = \mathbf{P}\mathbf{x}$$

$$\nabla \mathbf{p} = \mathbf{P} \text{ and } \nabla \mathbf{u} = \mathbf{P} - \mathbf{I}$$

$$\boldsymbol{\varepsilon} = \frac{1}{2} (\nabla \mathbf{u} + [\nabla \mathbf{u}]^T + [\nabla \mathbf{u}]^T \nabla \mathbf{u})$$

Simulation Loop

- **Compute Strain:**

$$\boldsymbol{\varepsilon} = \frac{1}{2} (\nabla \mathbf{u} + [\nabla \mathbf{u}]^T + [\nabla \mathbf{u}]^T \nabla \mathbf{u})$$

- **Convert to Stress:**

$$\boldsymbol{\sigma} = E \boldsymbol{\varepsilon}$$

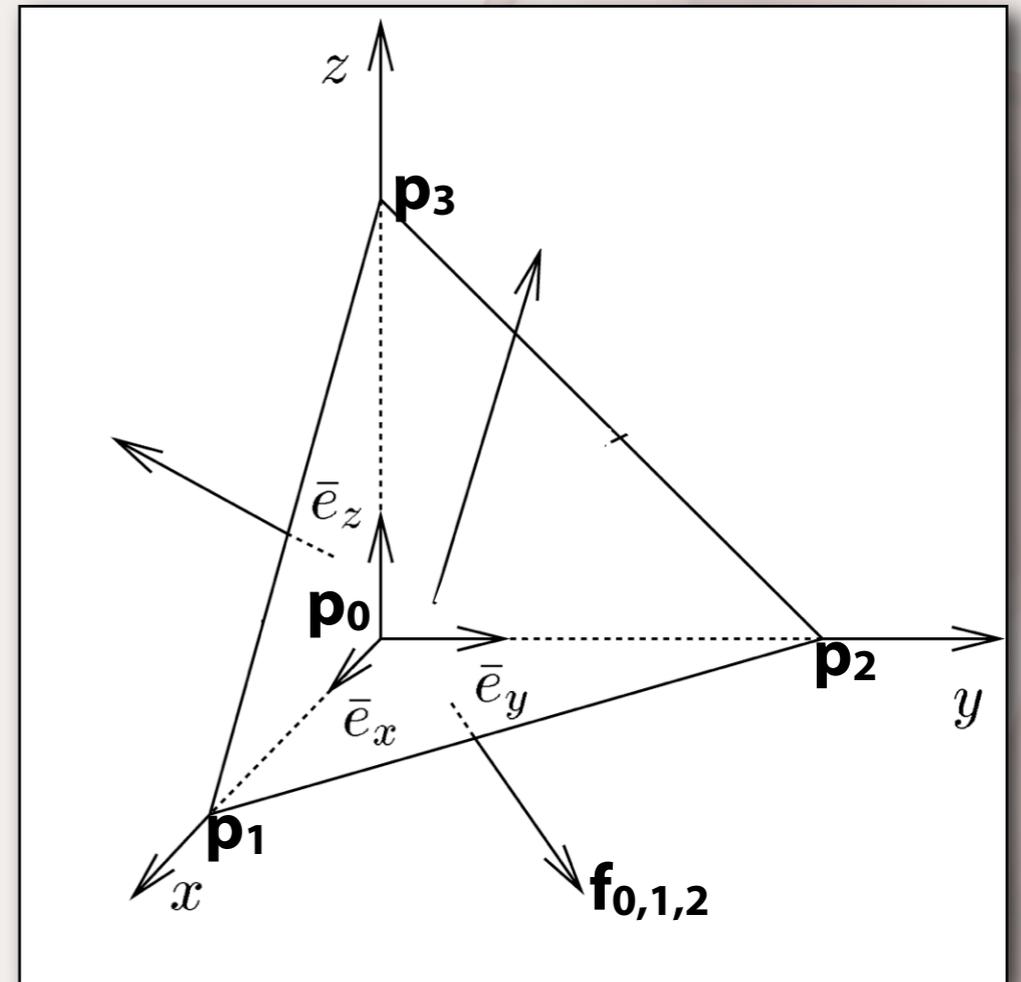
$$\frac{d\mathbf{f}}{dA} = \boldsymbol{\sigma} \cdot \mathbf{n}.$$

- **Compute Face Forces:**

$$\begin{aligned} \mathbf{f}_{0,1,2} &= \boldsymbol{\sigma} \cdot \mathbf{n}_{0,1,2} \cdot A_{0,1,2} \\ &= \boldsymbol{\sigma} [(\mathbf{p}_1 - \mathbf{p}_0) \times (\mathbf{p}_2 - \mathbf{p}_0)] \end{aligned}$$

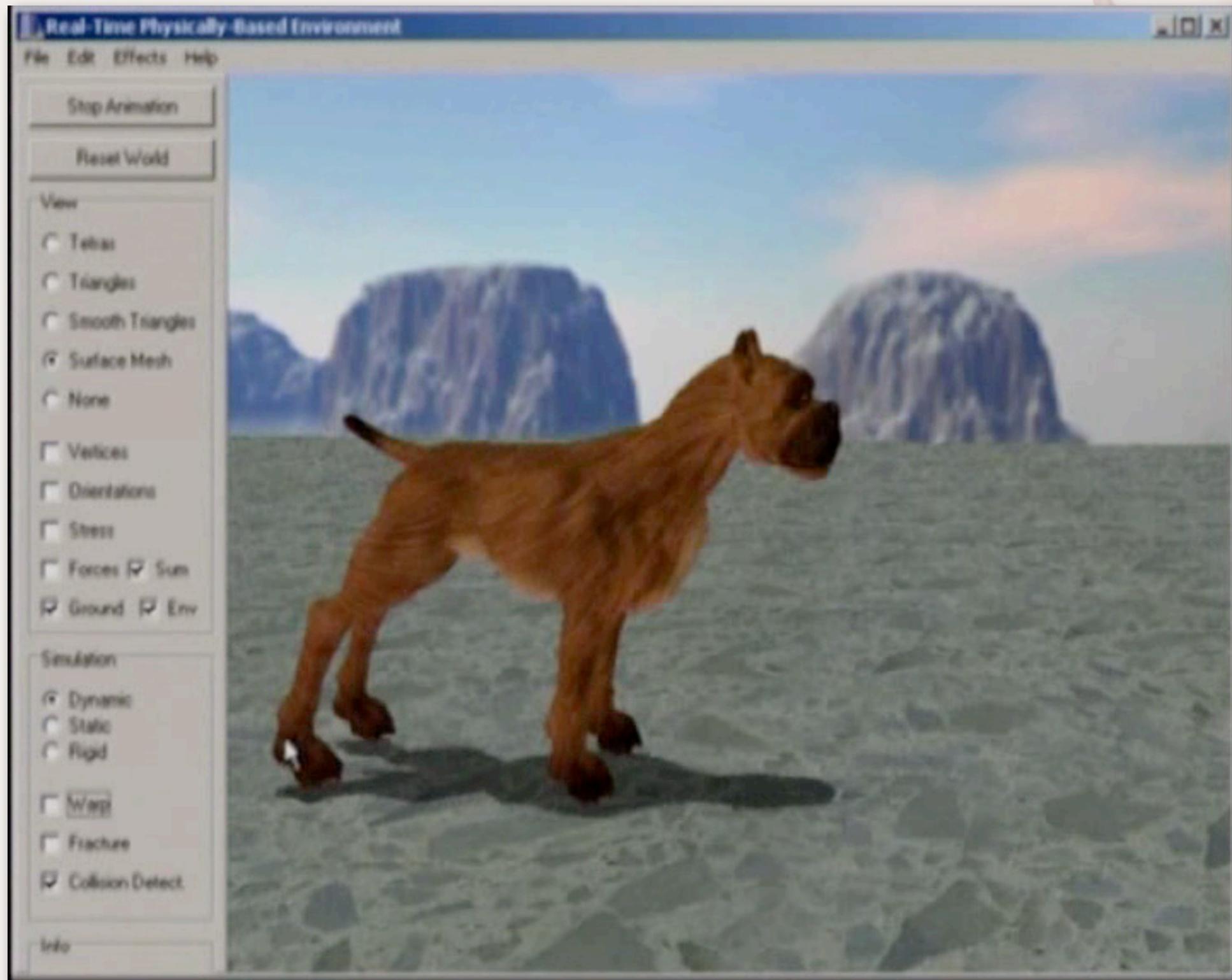
- **Distribute to vertices.**

- **Integrate eqns of motion** (e.g. 4th order RK).



source: <http://www.emeraldinsight.com/fig/1740200102032.png>

Examples



Question...

- Why does the strain matrix have only 6 degrees of freedom?

