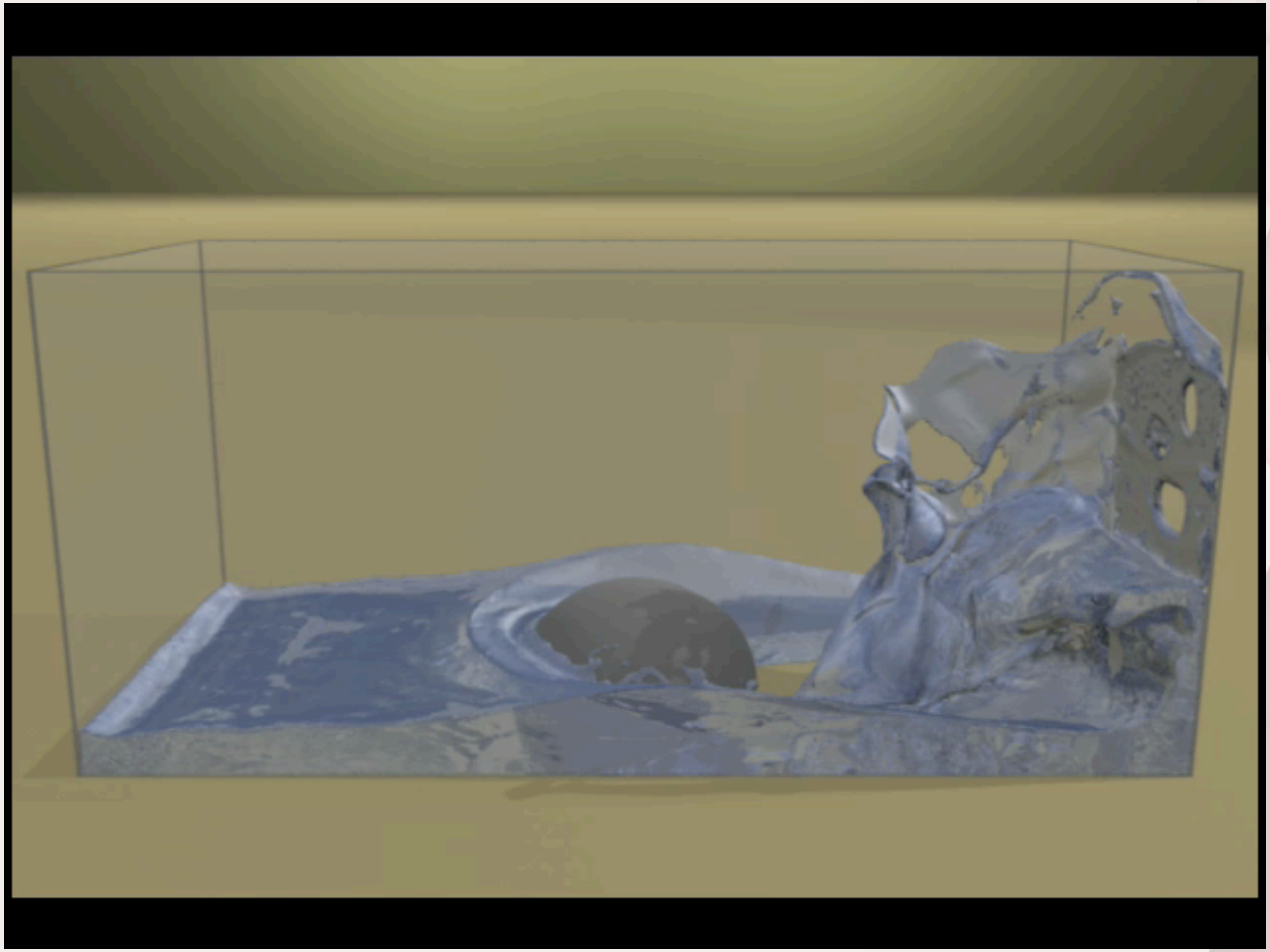
Free Surface Fluids

Adrien Treuille



source: Chentanez *et al* [2007]

Overview

- **Questions about project 2?**
- **Solid Boundaries.**
  - ***Affect on the advection step?***
  - ***Affect on the projection step?***
- **Free-surfaces.**
  - ***Affect on the advection step?***
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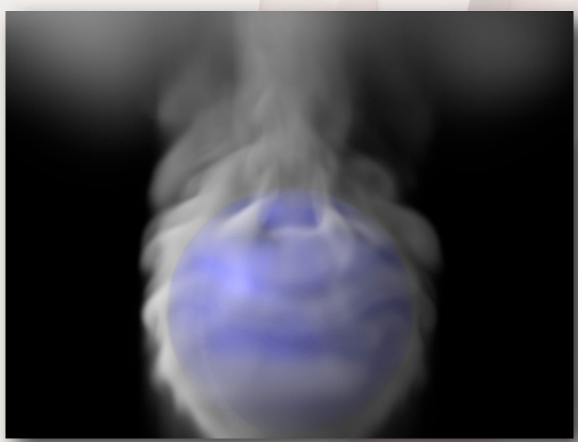
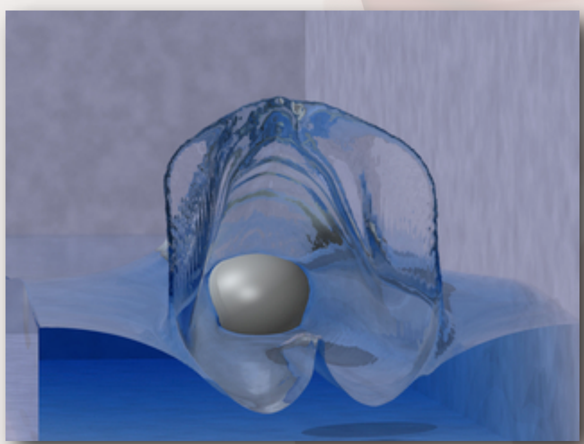
- **Open Challenges**
- **Closing Statements**





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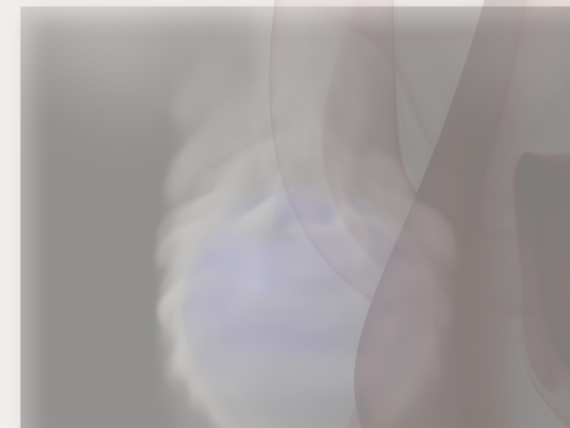

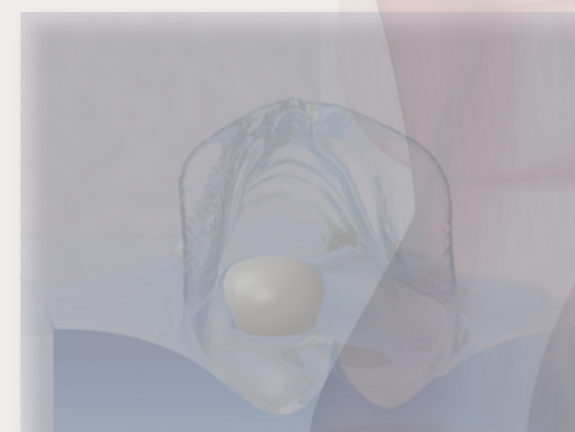

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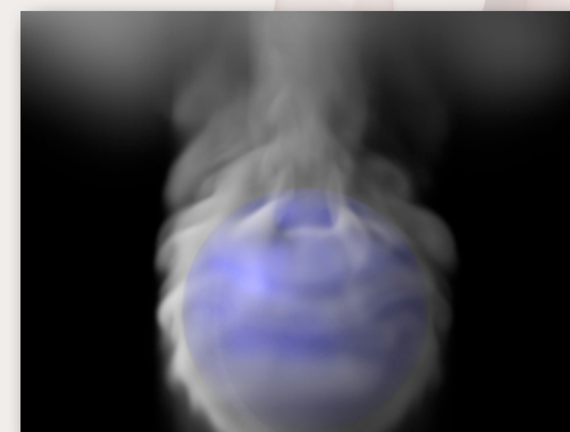

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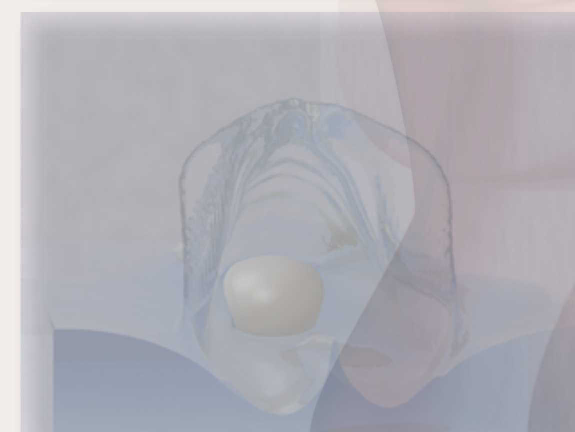

- Open Challenges
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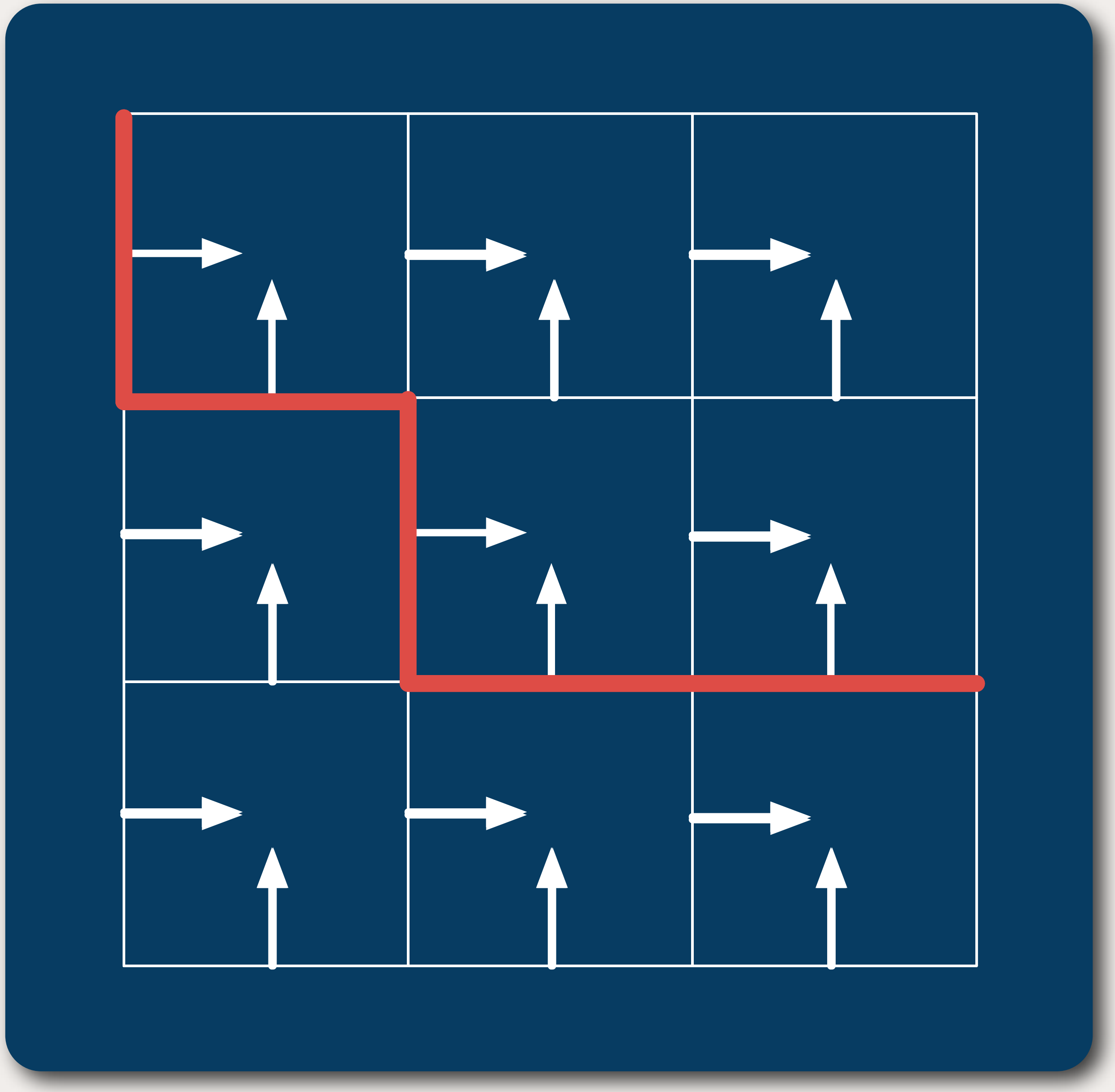
Solid Boundaries

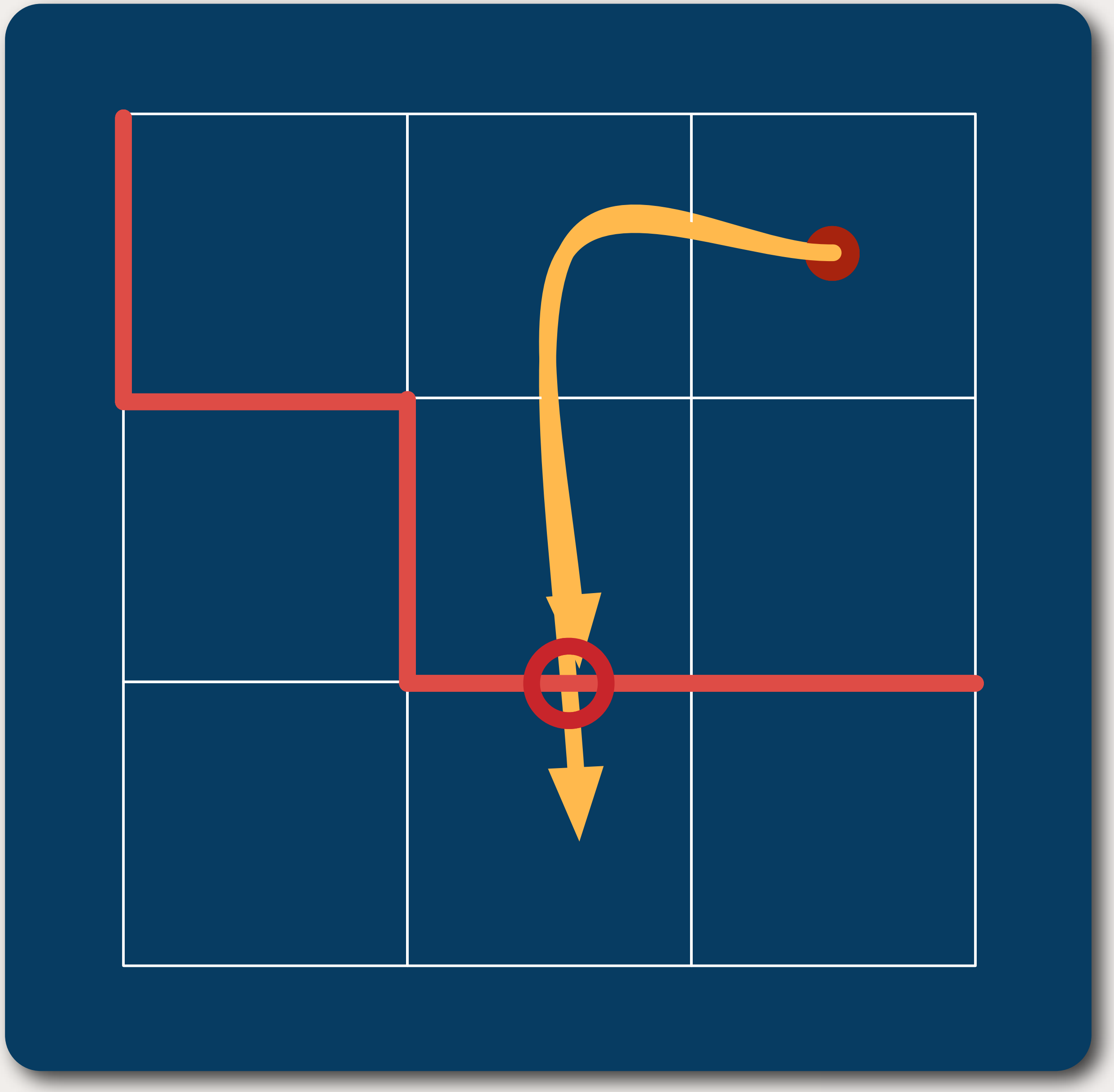


Condition:

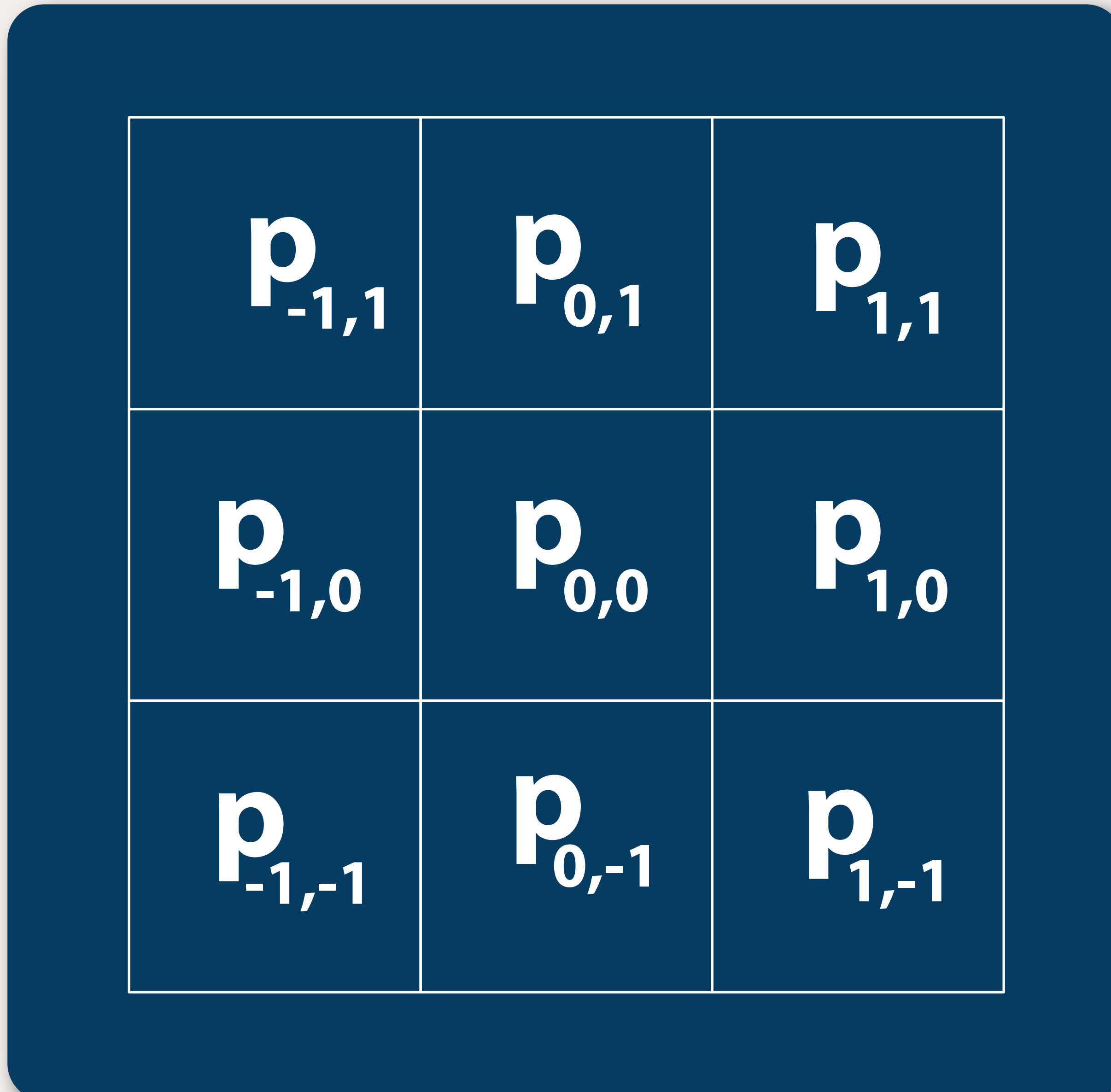
$$\mathbf{u} \cdot \mathbf{n} = 0$$

- **How does this affect advection?**
- **How does this affect projection?**

Path Clipping



Pressure



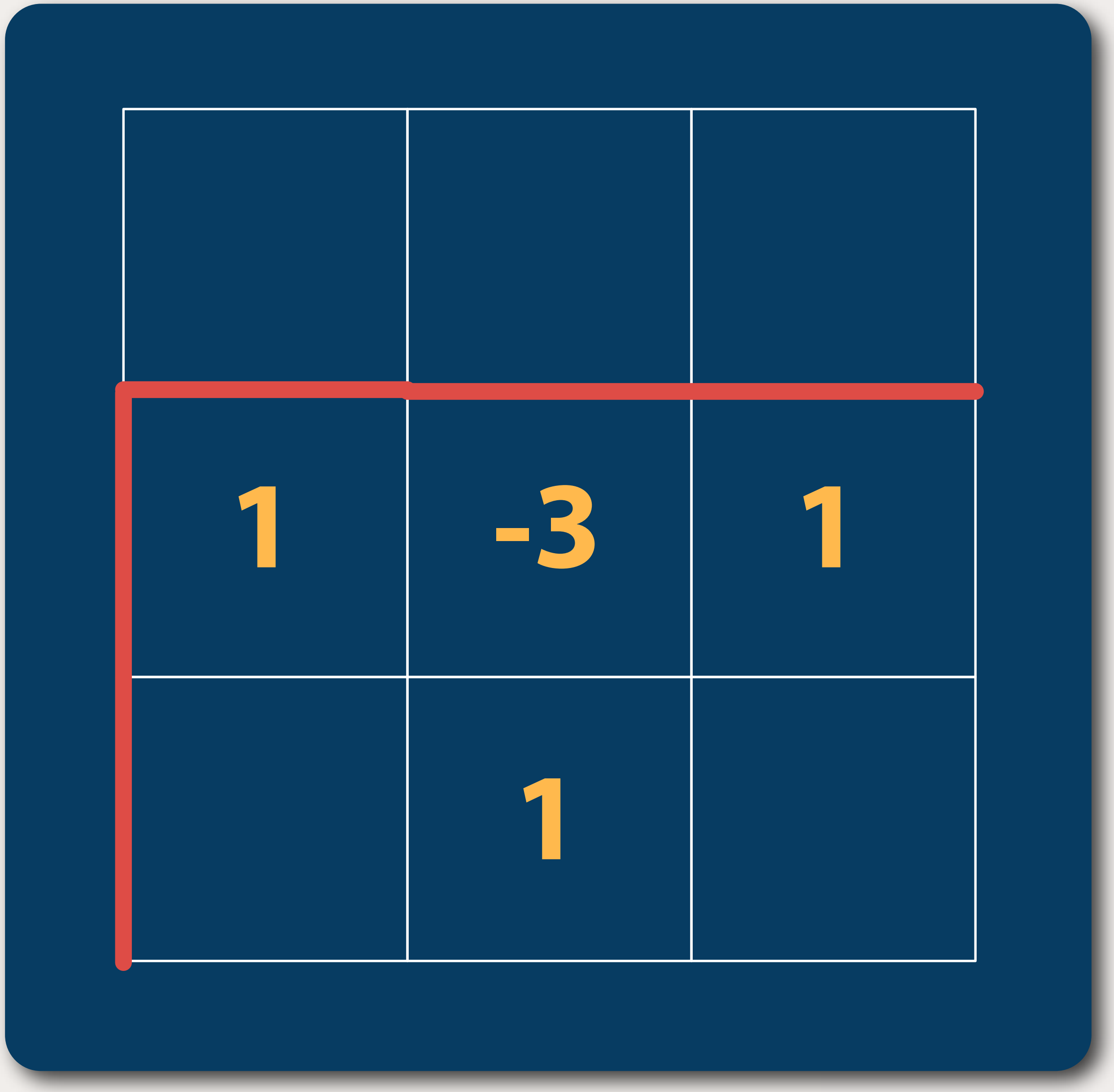
$$\nabla \mathbf{u}_{0,0} = p_{0,-1} + p_{0,1} + p_{-1,0} + p_{1,0} + 4p_{0,0}$$

Pressure

	1	
1	-4	1
	1	

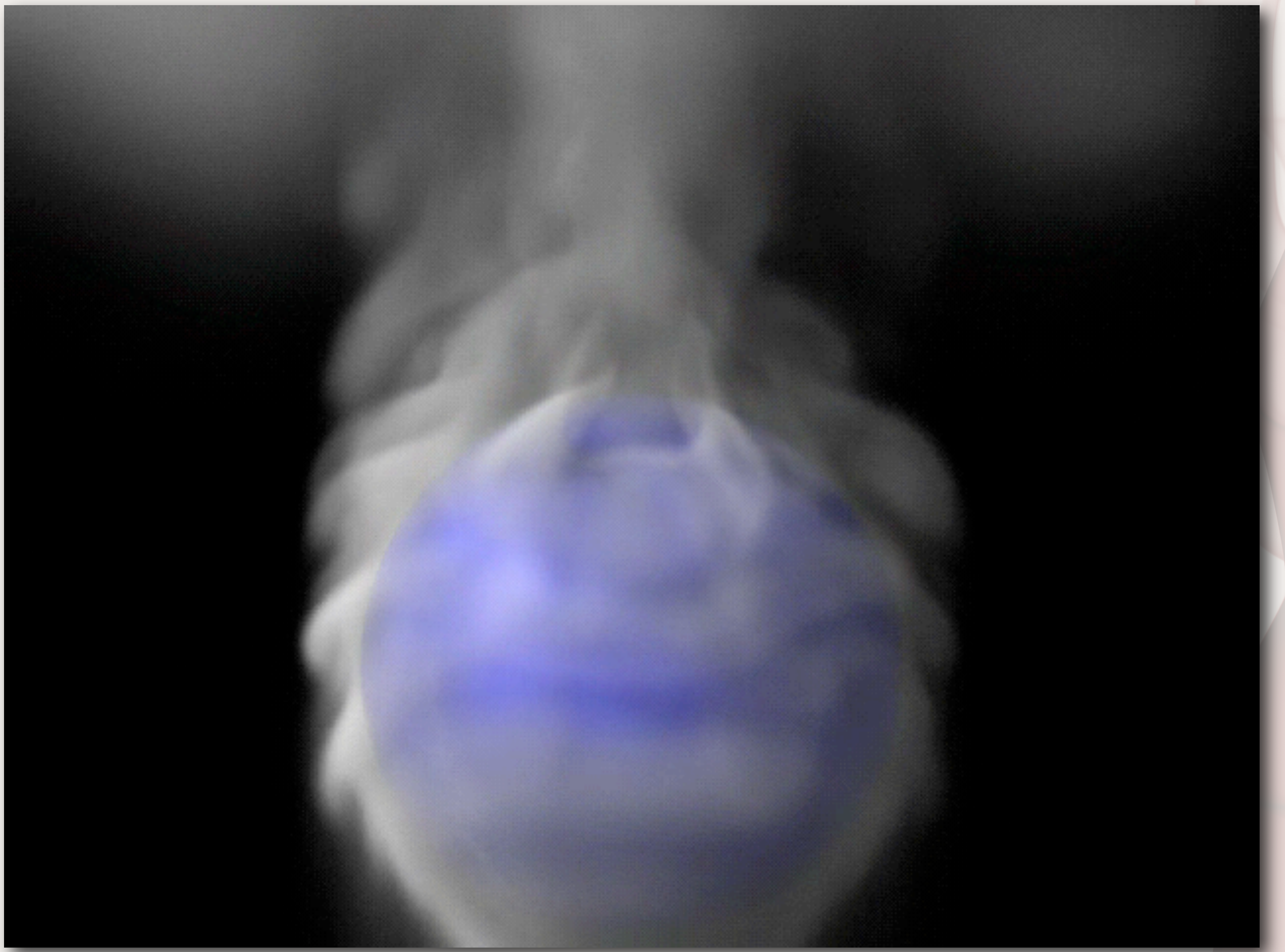
$$\nabla \mathbf{u}_{0,0} = p_{0,-1} + p_{0,1} + p_{-1,0} + p_{1,0} + 4p_{0,0}$$

Pressure



$$\nabla \mathbf{u}_{0,0} = p_{0,-1} + p_{0,1} + p_{1,0} - 3p_{0,0}$$

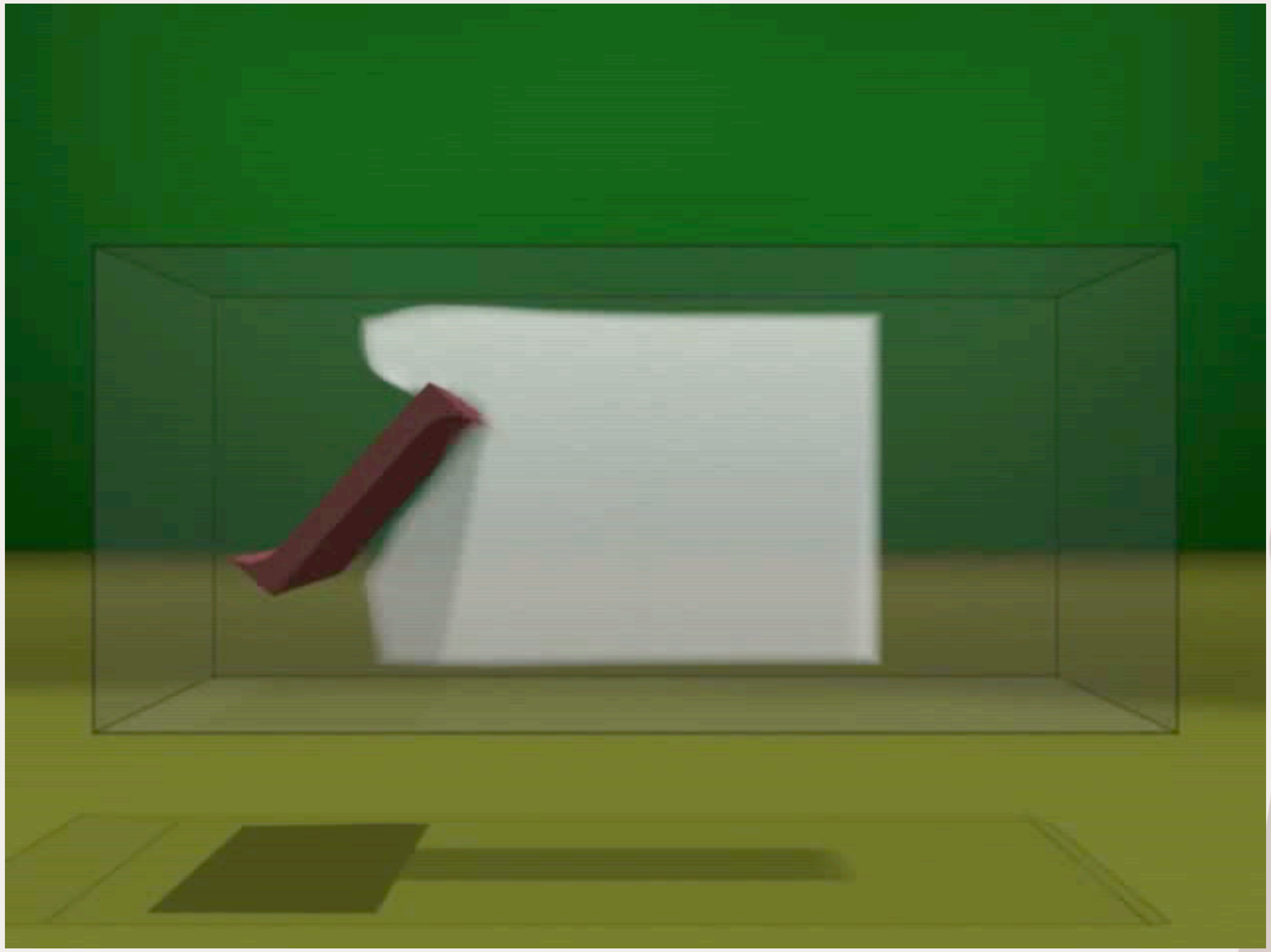
Example



source: Losasso, Gibou, and Fedkiw [2004]

Question

- **What about non-rectilinear boundaries?**
  - **Tetrahedral meshes.**



source: Feldman O'Brien and Klingner [2005]

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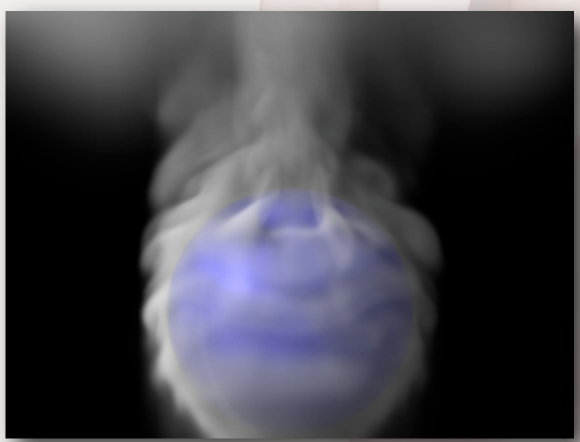

- Free-surfaces.
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- Open Challenges
- Closing Statements

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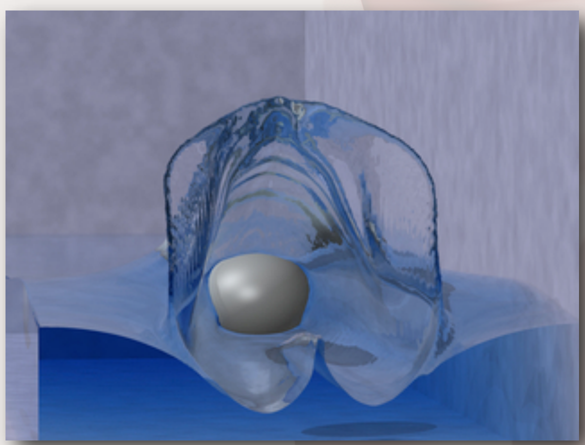
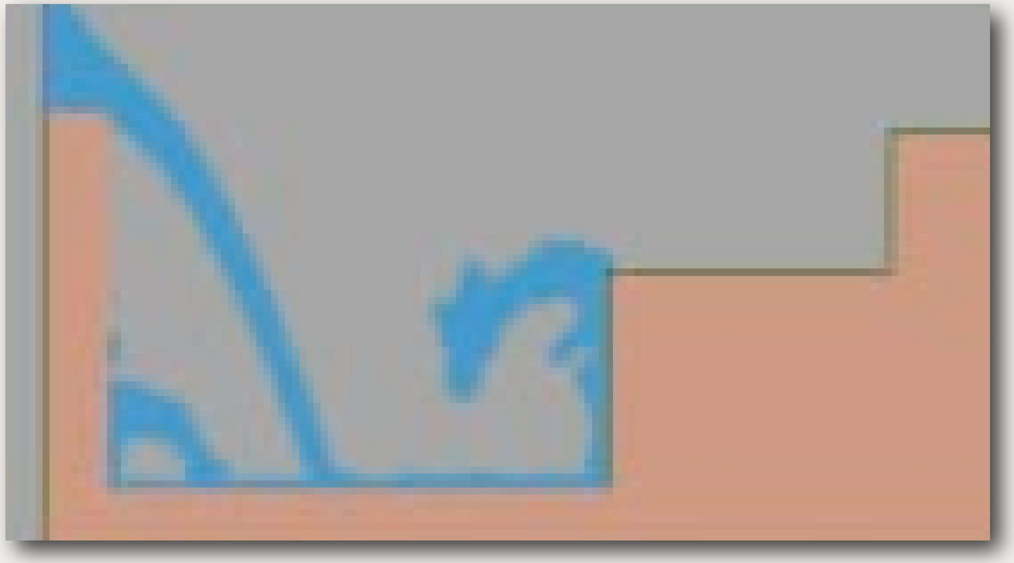


Free Surfaces

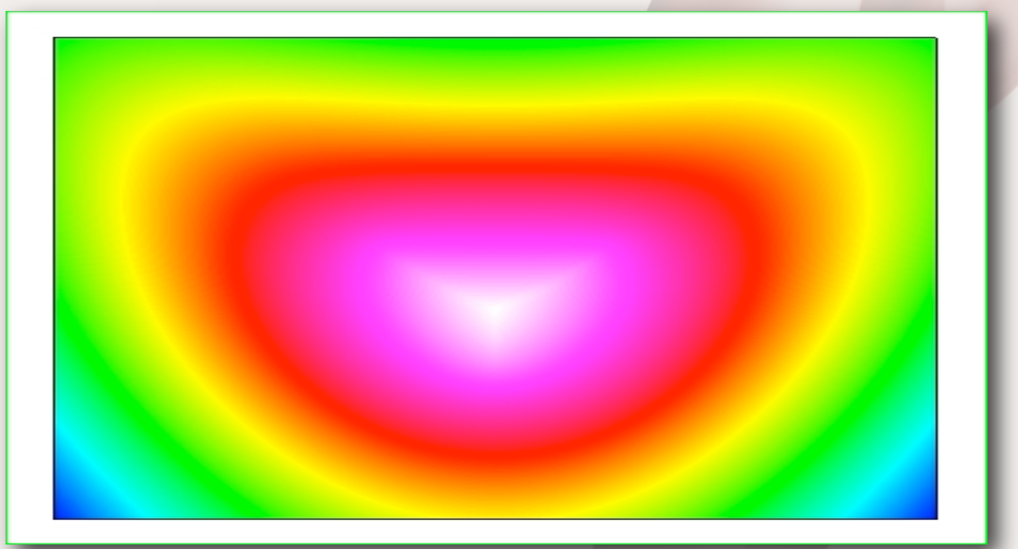
- **Surface between two fluids.**

source: http://plus.maths.org/issue22/news/skimming/

Volume of Fluids

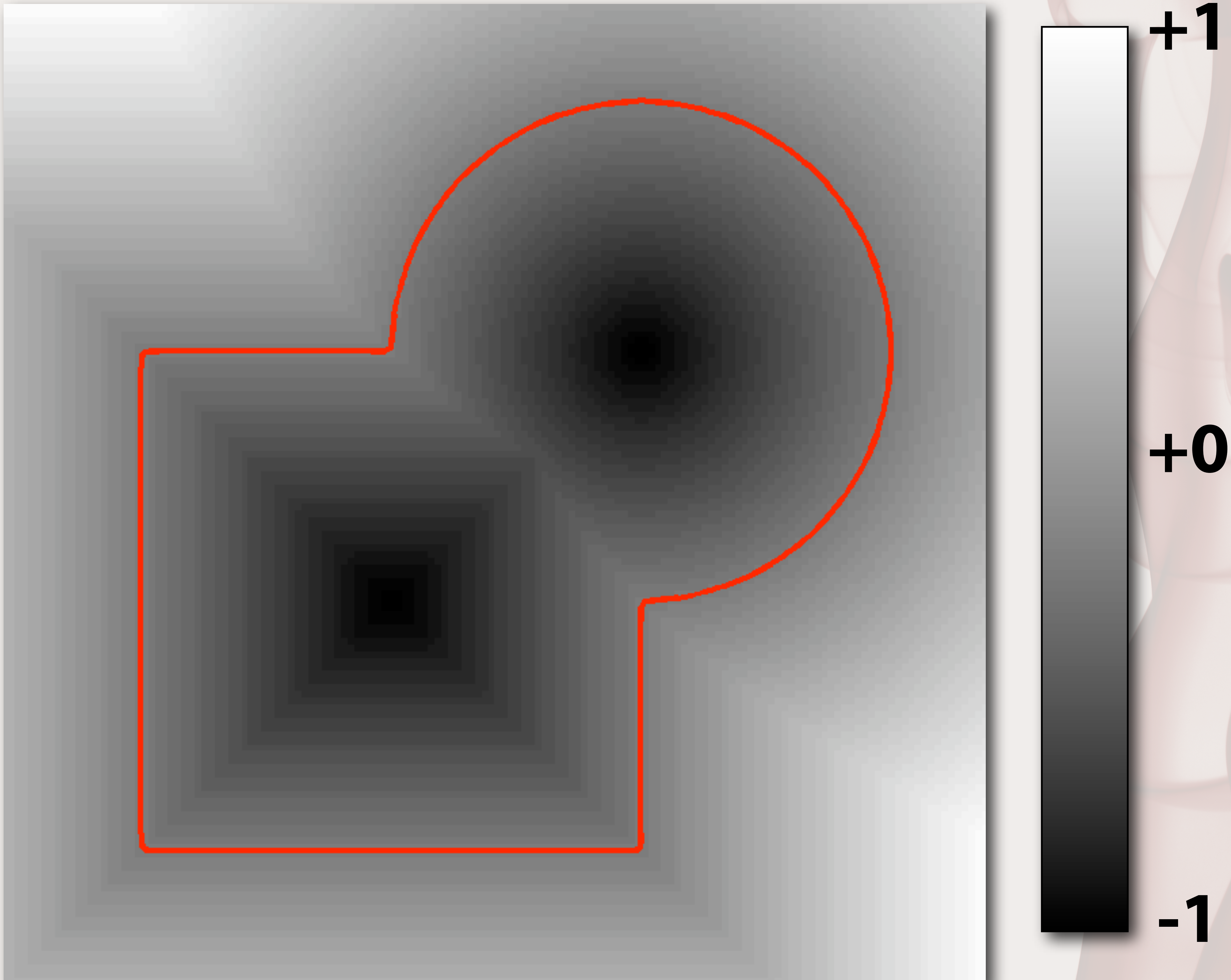


Signed Distance Function



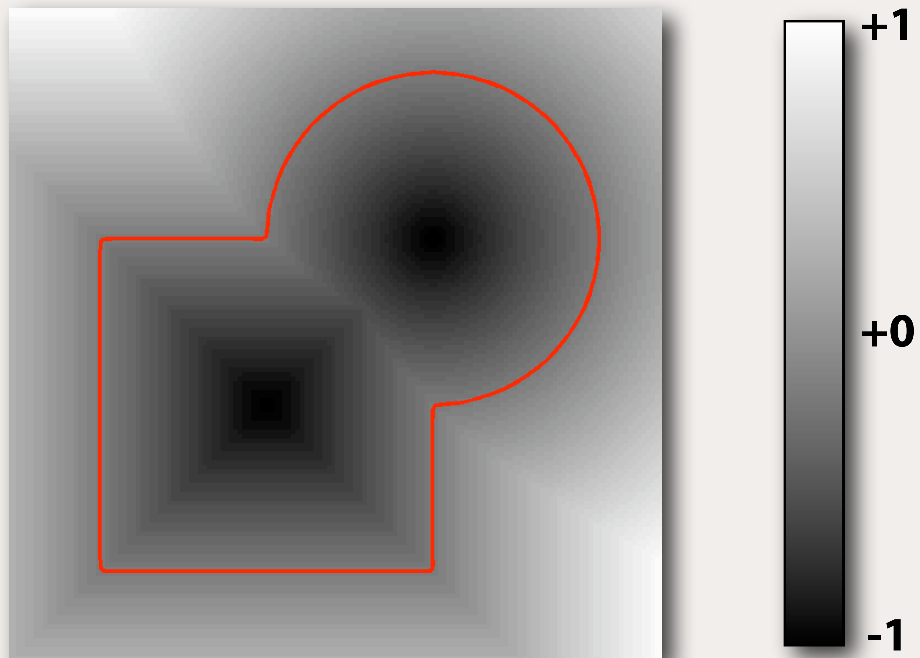
source: http://www.csc.fi/english/pages/elmer/examples/fallingdrop/

Signed Distance Function



source: http://www.ceremade.dauphine.fr/~peyre/cours/manifold/

Signed Distance Function



source: http://www.ceremade.dauphine.fr/~peyre/cours/manifold/

- **Easy to know where water is.**
- **Good surface reconstruction: marching cubes algorithm.**
- **Advection OK!**
- **Must be redistanced:**

$$\phi\big|_{\partial M} = 0 \quad ||\nabla \phi|| = 1$$

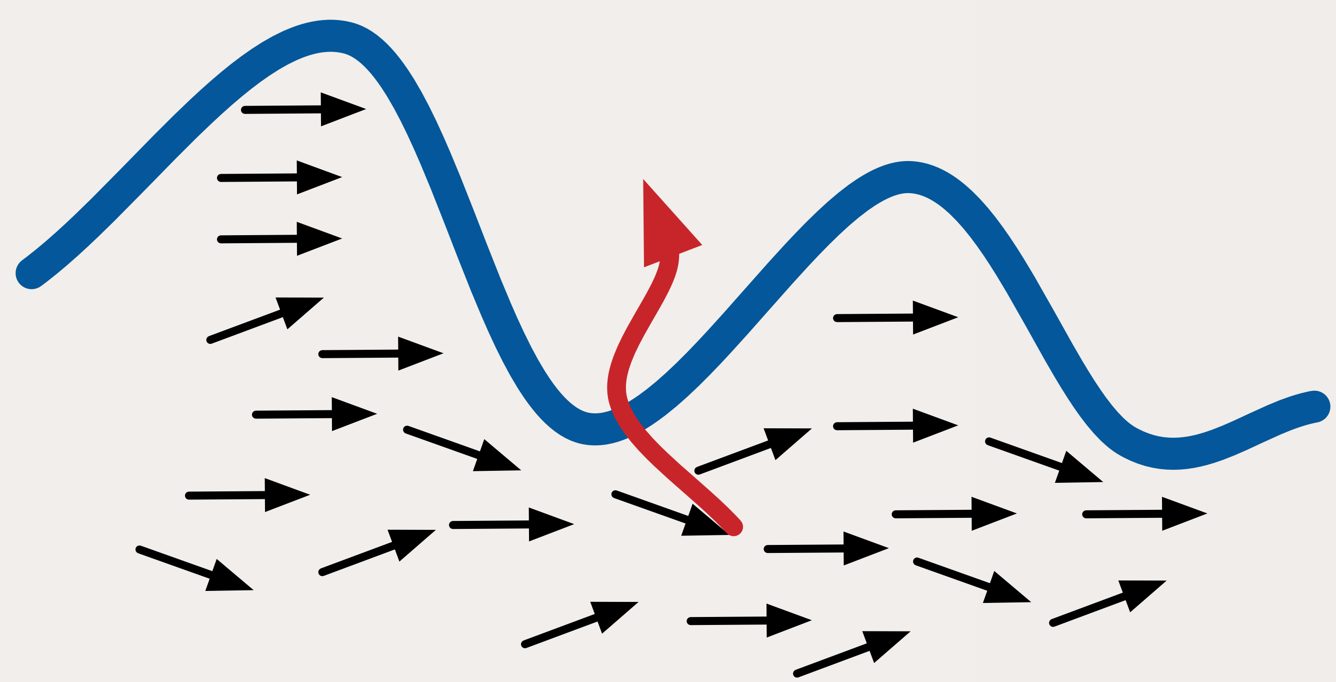
Signed Distance Questions

- How can we perform intersection?
- How can we perform union?

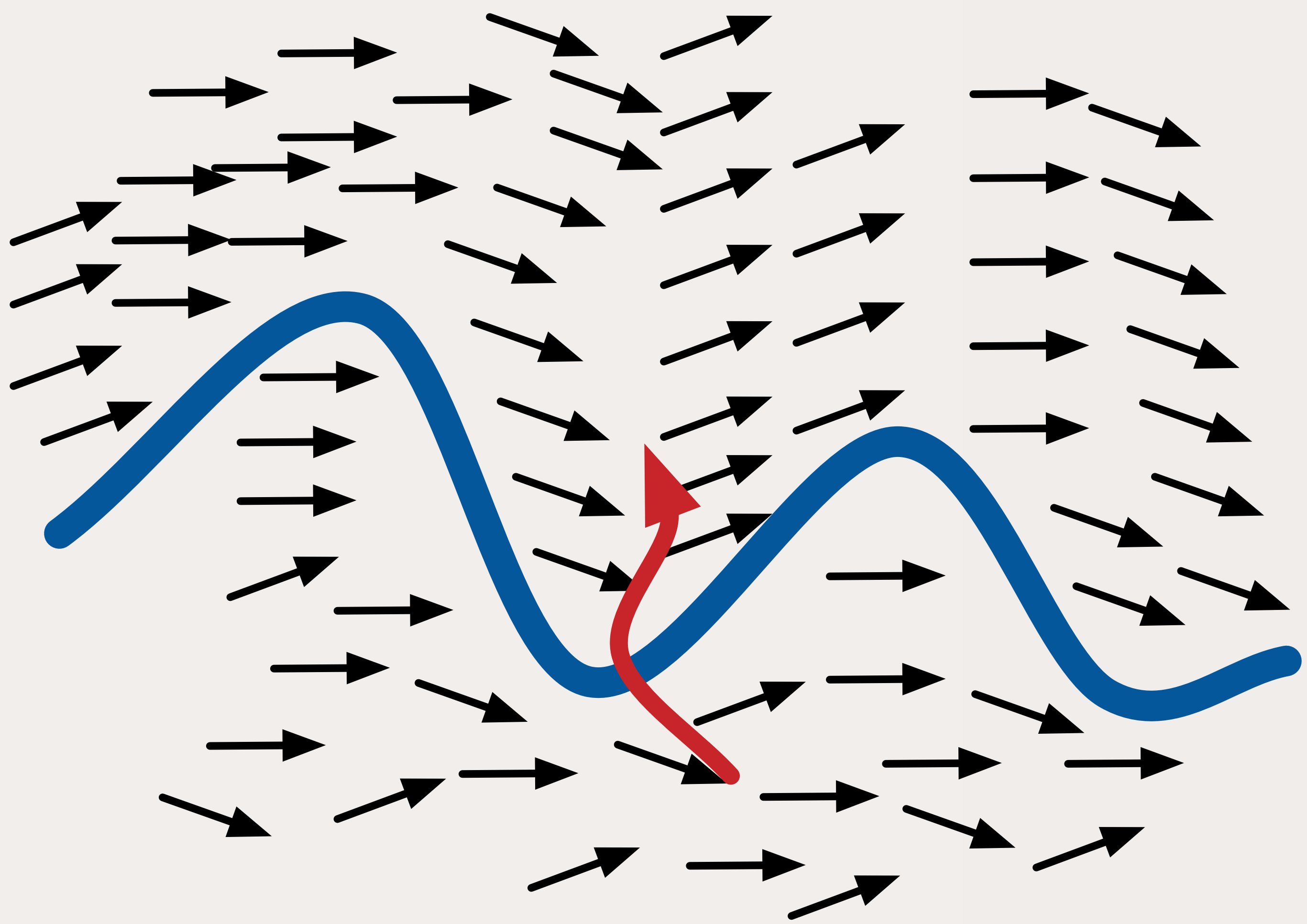
Liquid Simulation Issues

- How do we change the advection step?
- How do we change the projection step?

Path Clipping



Velocity Extension



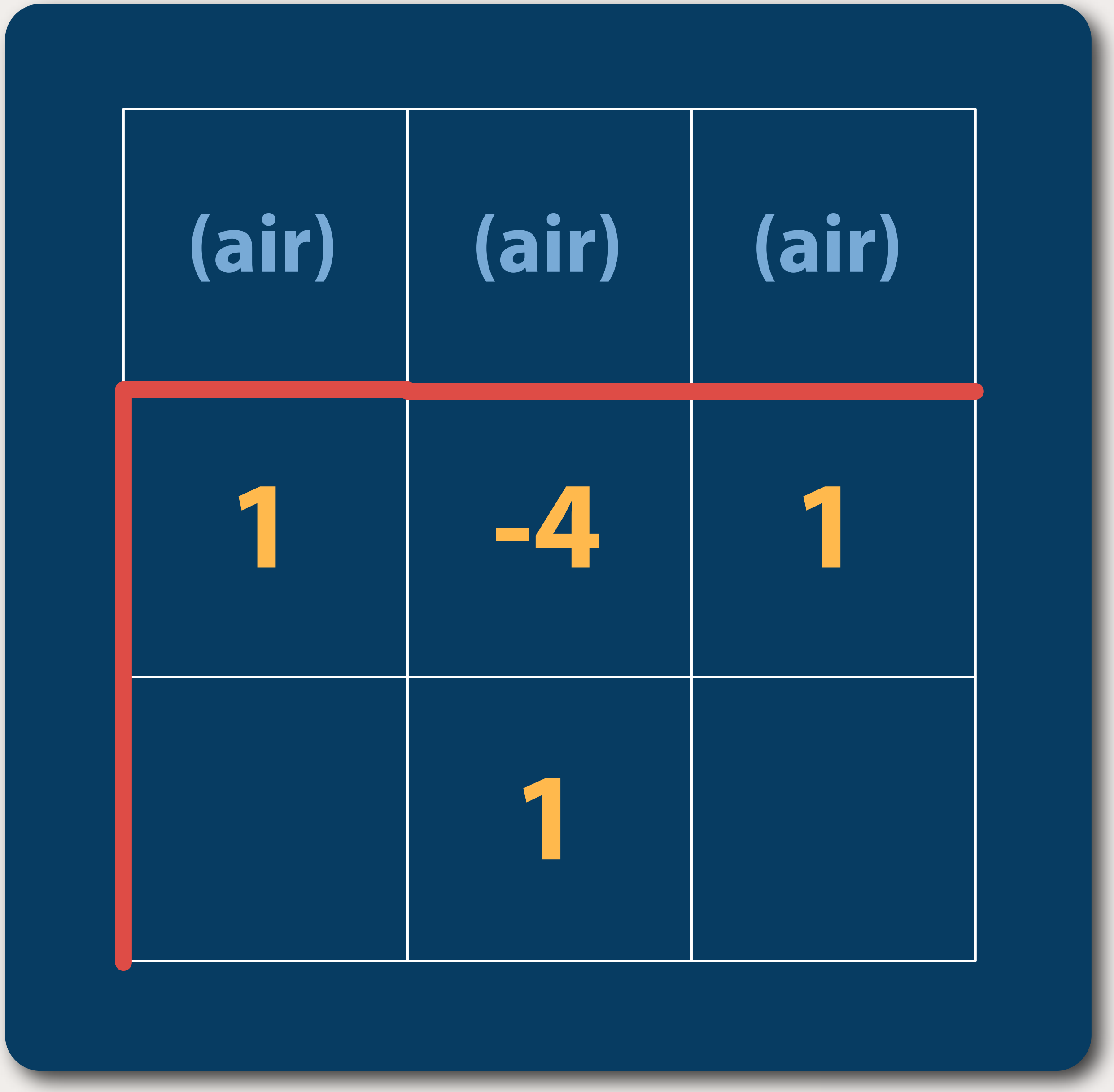
D. ADALSTEINSSON AND J. A. SETHIAN. *The Fast Construction of Extension Velocities in Level Set Methods*. Journal of Computational Physics [1999]

Pressure

	1	
1	-4	1
	1	

$$\nabla \mathbf{u}_{0,0} = p_{0,-1} + p_{0,1} + p_{-1,0} + p_{1,0} + 4p_{0,0}$$

Pressure



$$\nabla \mathbf{u}_{0,0} = p_{0,-1} + p_{0,1} + p_{1,0} - 4p_{0,0}$$

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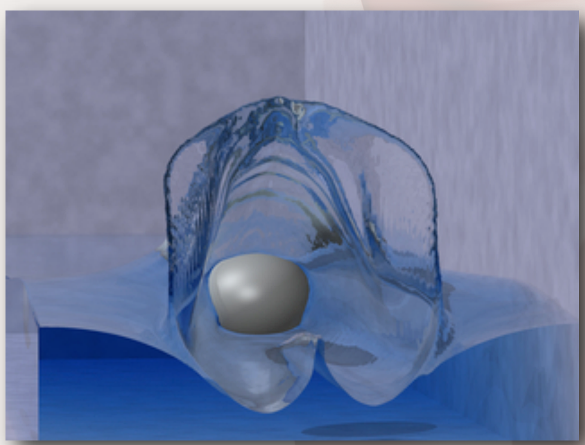

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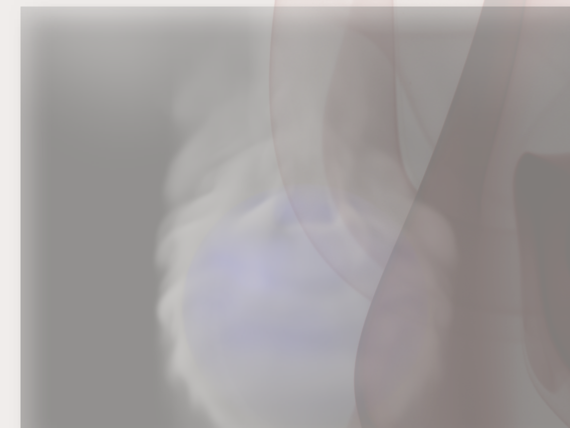

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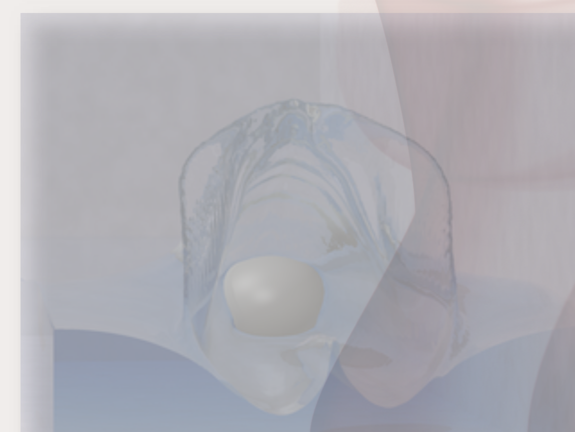

- **Open Challenges**
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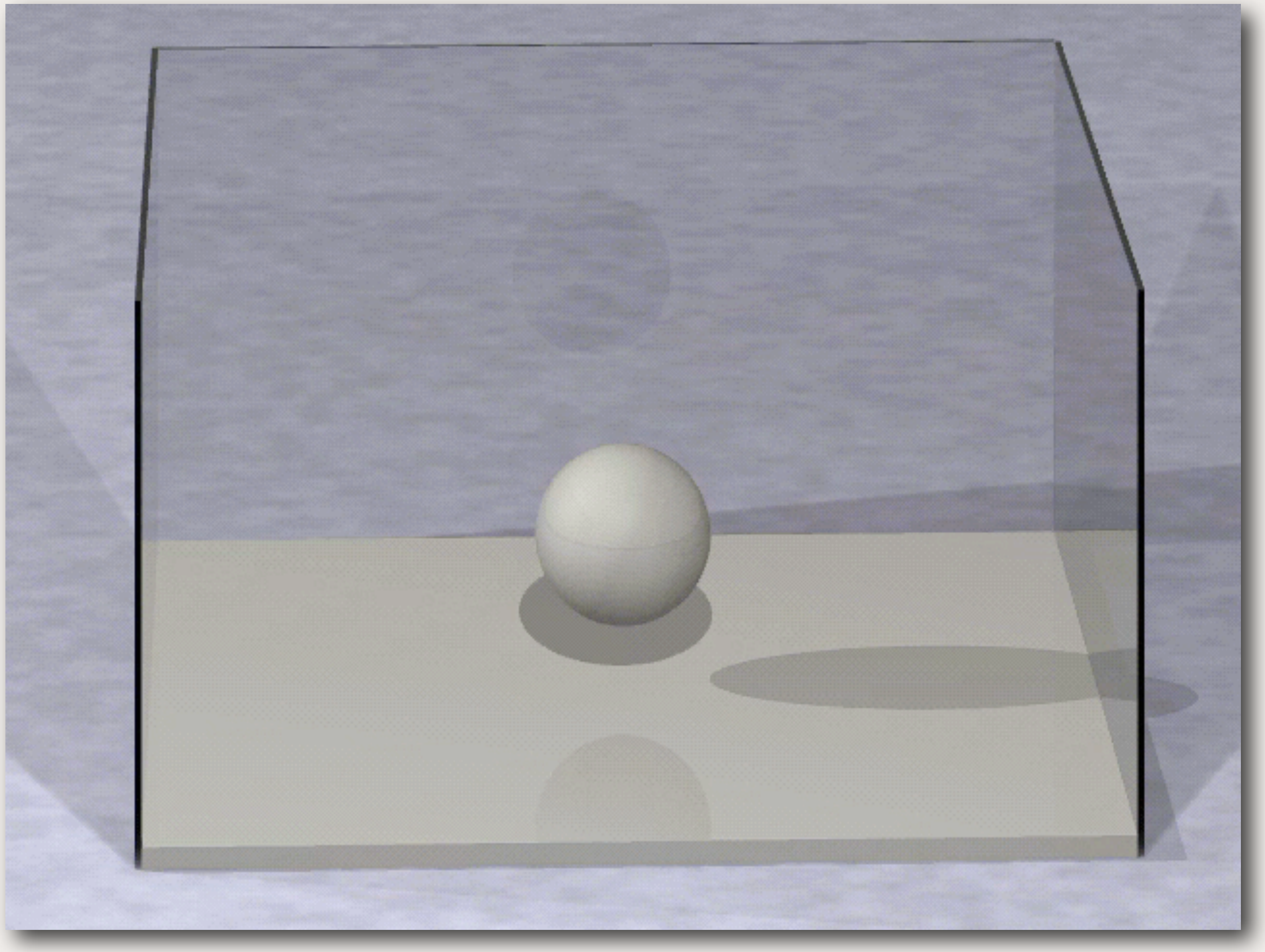
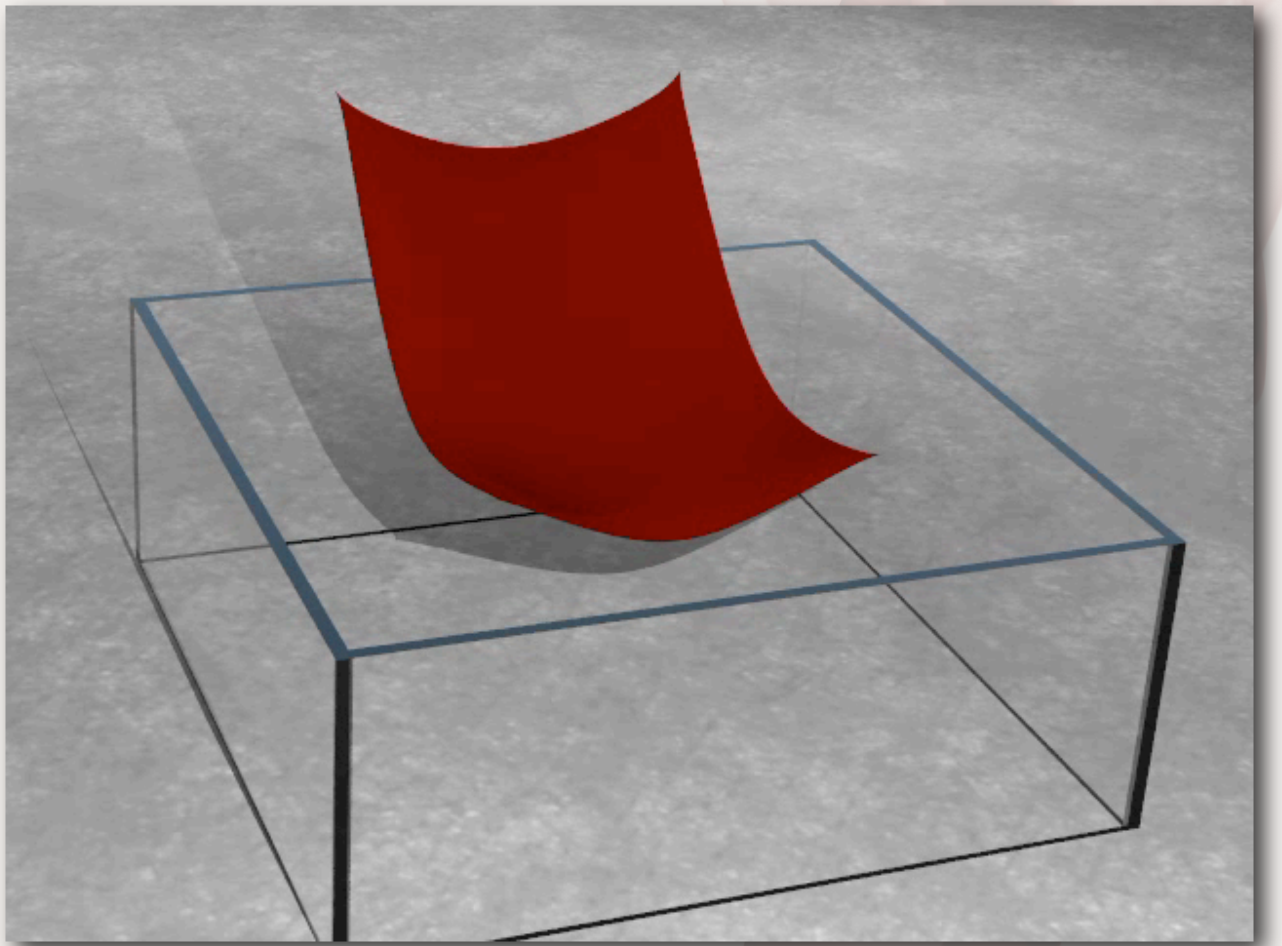
Resolving Small Features

Quad Trees



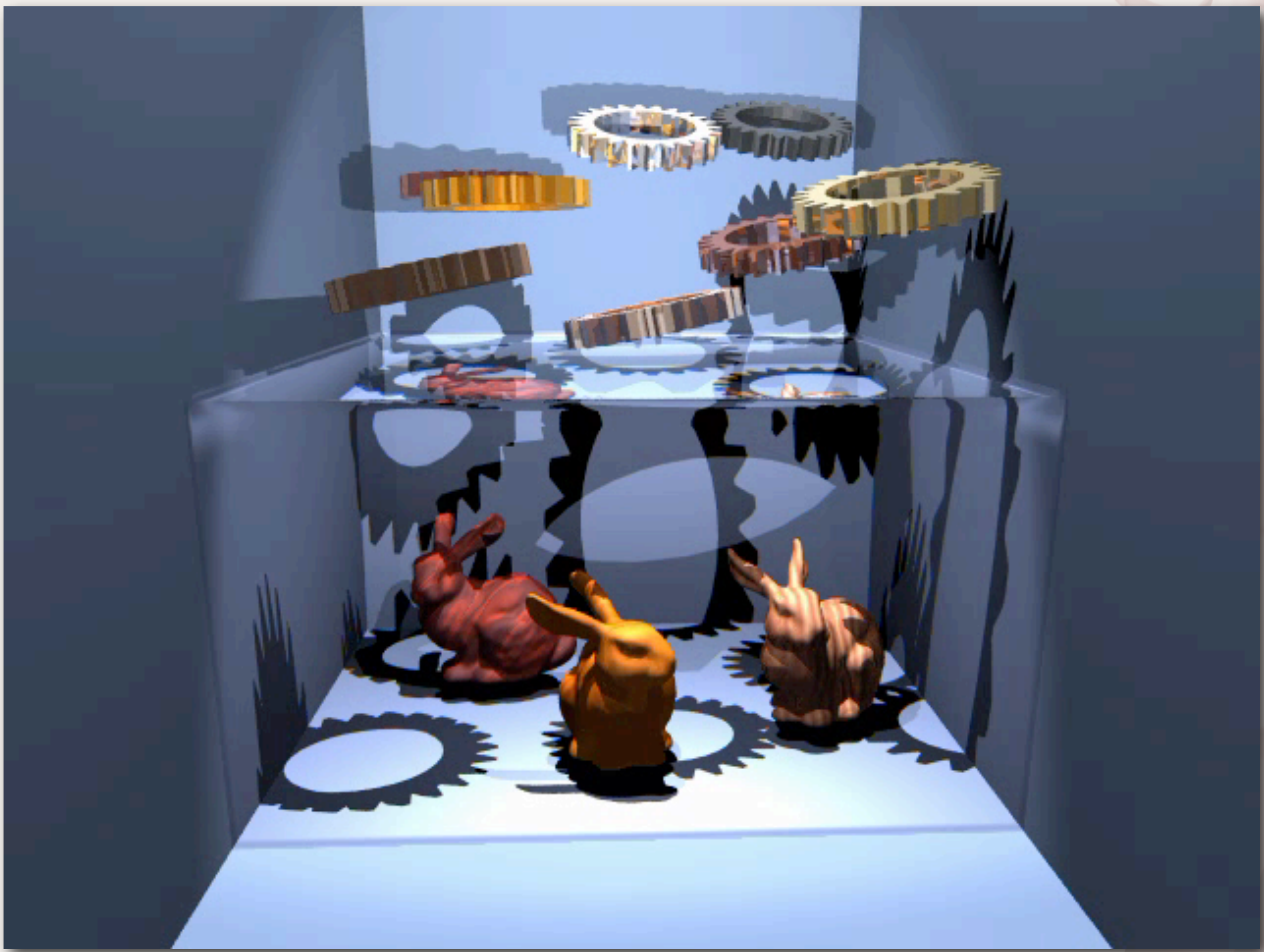
source: Losasso, Gibou, and Fedkiw [2004]

Particles



source: Guendelman et. al. [2005]

Coupling



source: Carlson, Mucha, and Turk [2004]

Real-time

source: Treuille, Lewis, and Popović [2004]

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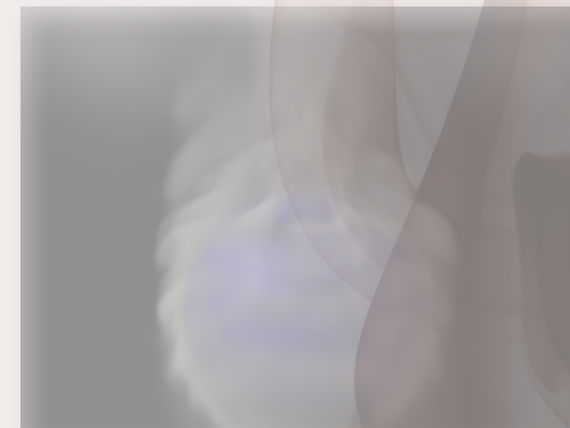

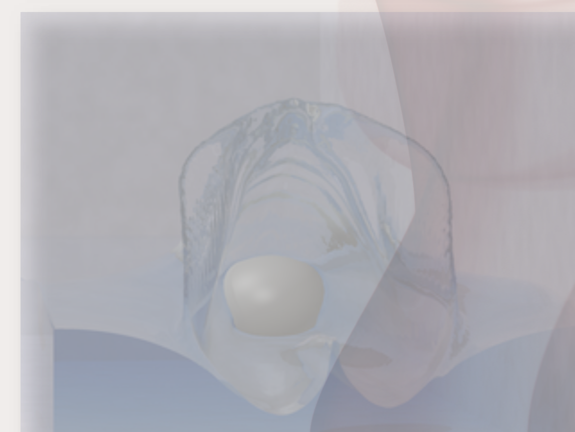

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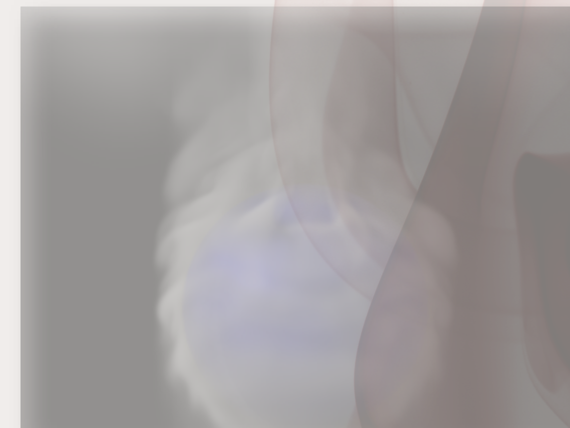
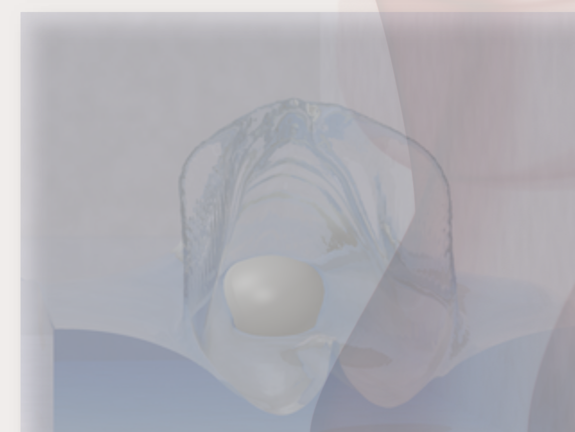




Closing Statements

- Next Wednesday's class.
- Question:
  - How can we preserve volume?

Rigid Body Dynamics

treuille@cs.cmu.edu

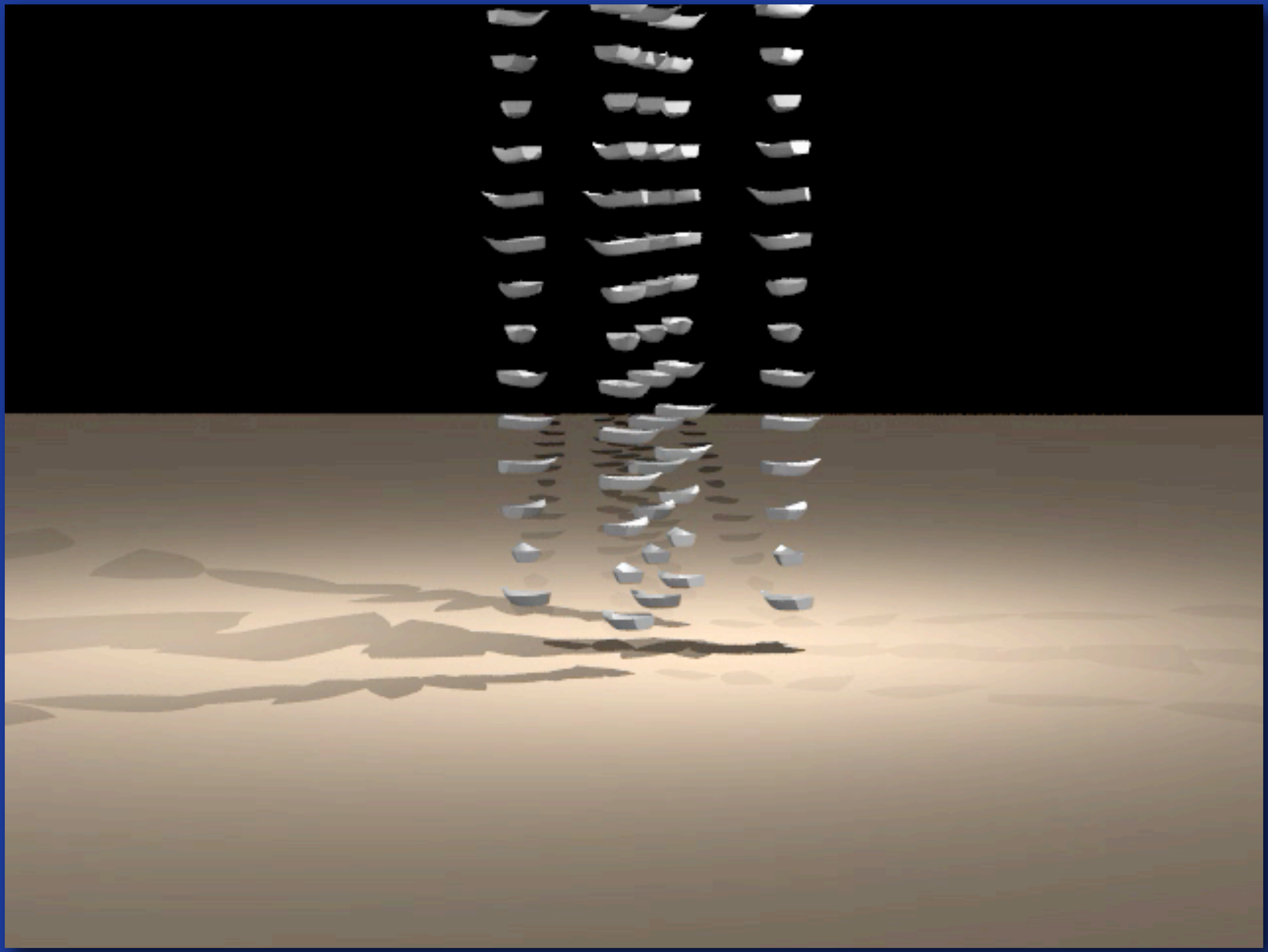
Rigid Body Dynamics

Rigid Body Dynamics

David Baraff

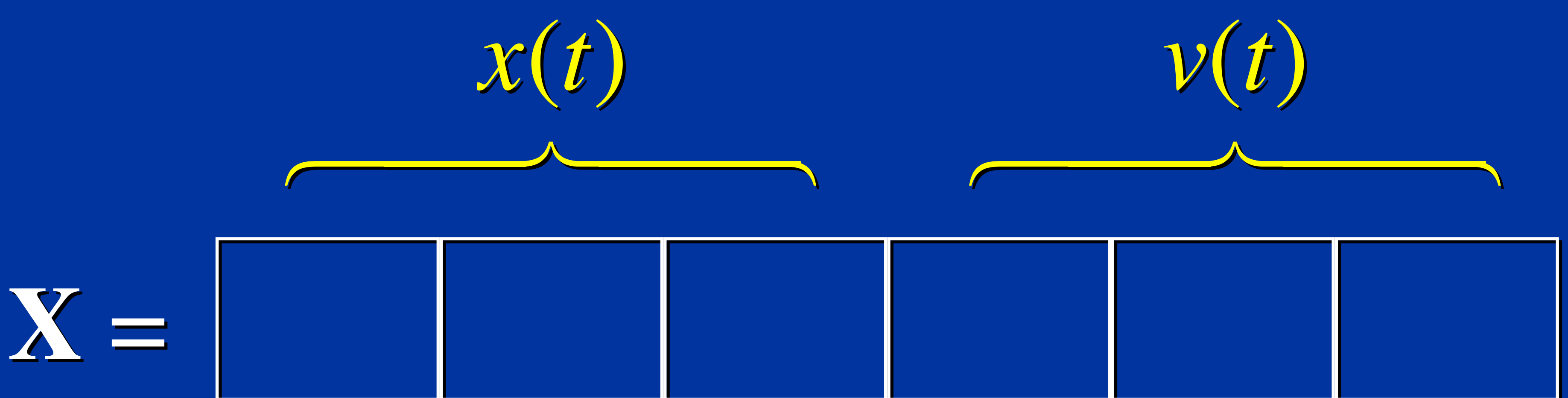
PIXAR
ANIMATION STUDIOS

What is a Rigid Body?



Particle State

$$\mathbf{X}(t) = \begin{pmatrix} x(t) \\ v(t) \end{pmatrix}$$



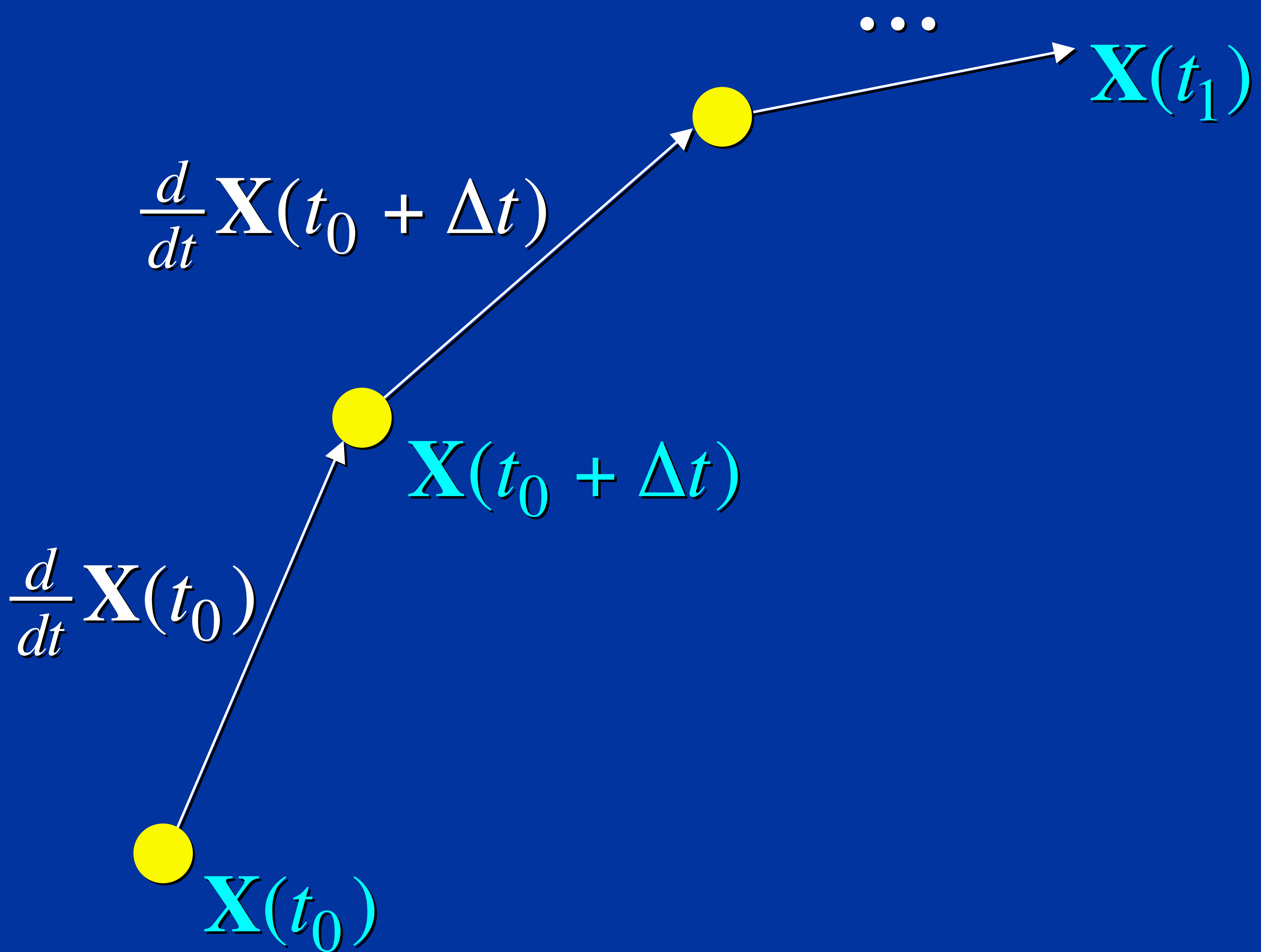
State Derivative

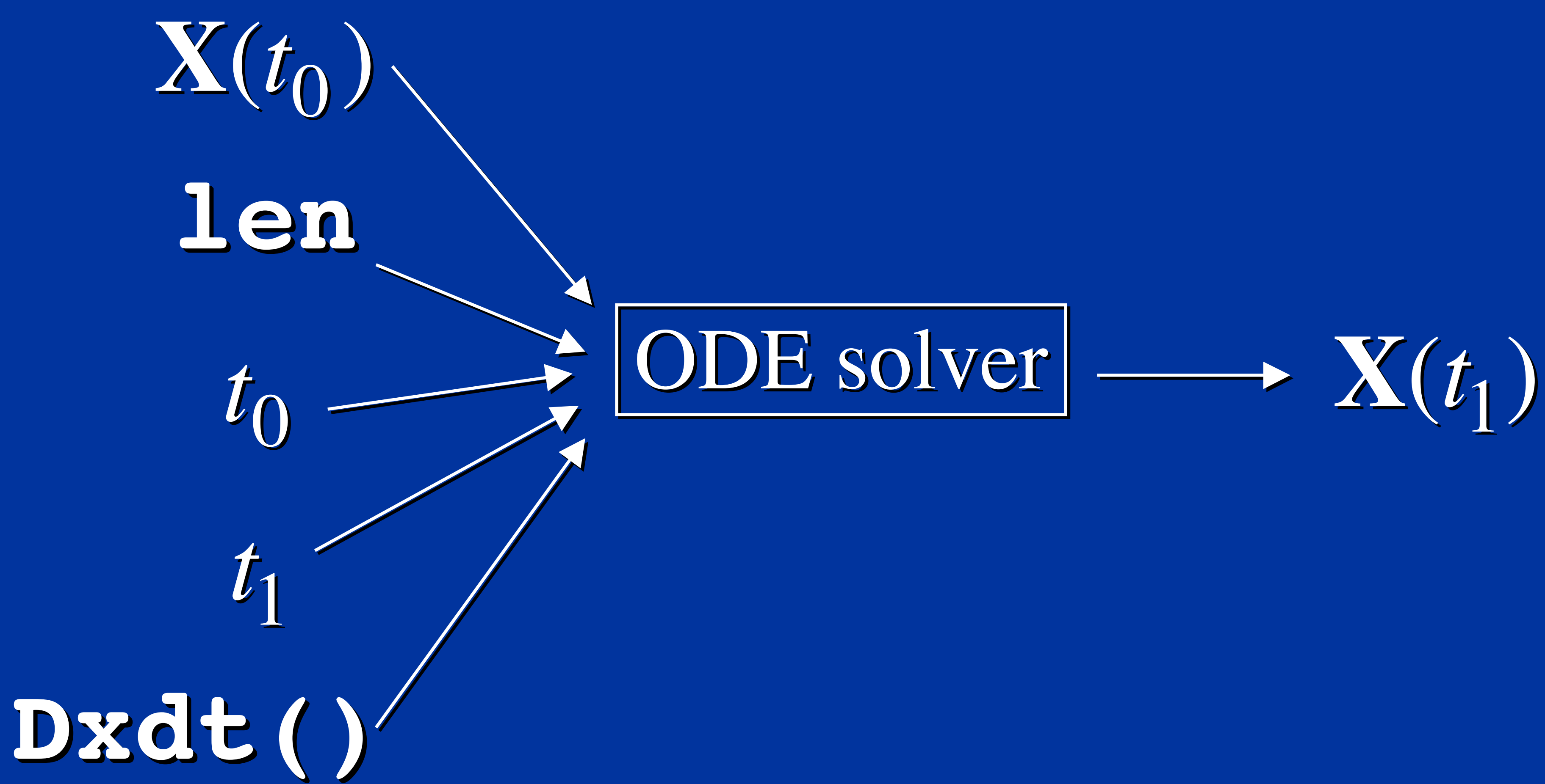
$$\frac{d}{dt}\mathbf{X} = \frac{d}{dt}\begin{pmatrix} x_1(t) \\ v_1(t) \\ \vdots \\ x_n(t) \\ v_n(t) \end{pmatrix} = \begin{pmatrix} v_1(t) \\ F_1(t)/m_1 \\ \vdots \\ v_n(t) \\ F_n(t)/m_n \end{pmatrix}$$

$$\frac{d}{dt}\mathbf{X} =$$

		$\cdots$ $6n$ elements $\cdots$		

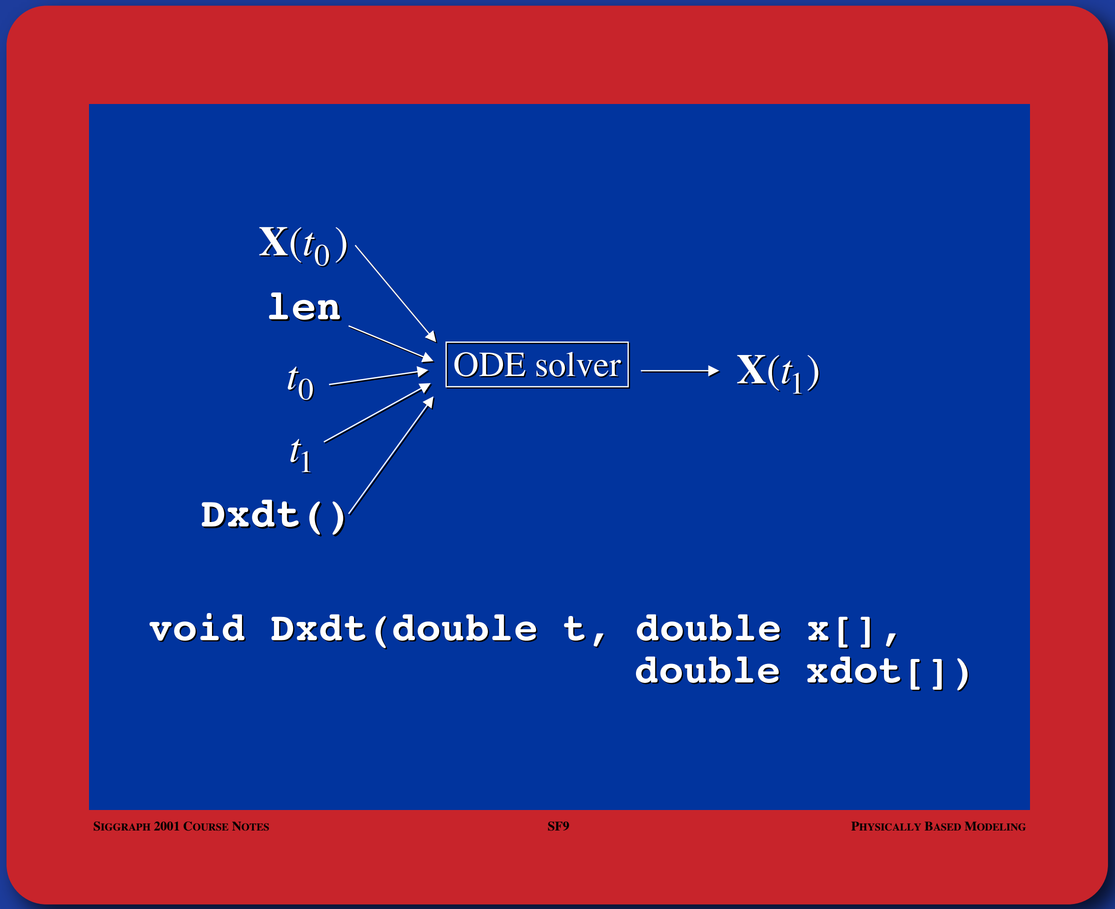
ODE solution



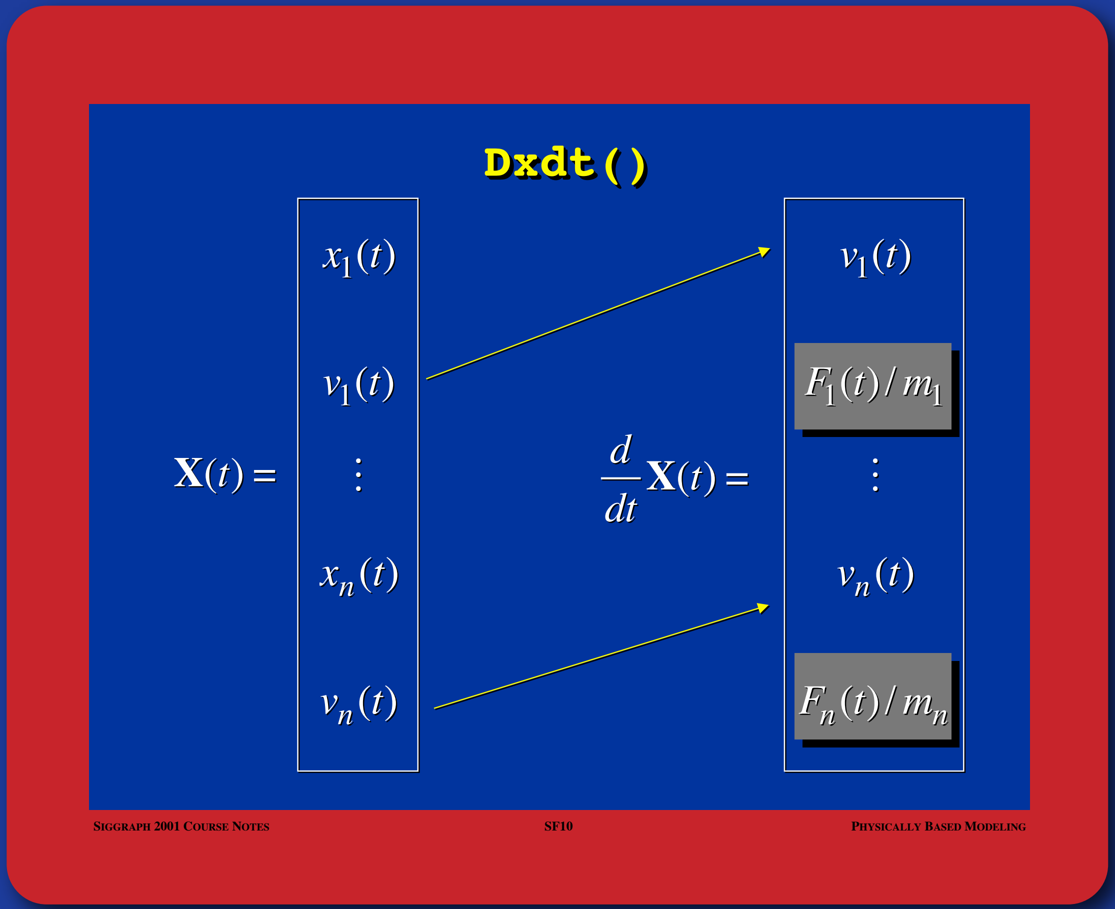



```
void Dxdt(double t, double x[],
                    double xdot[])
```

What We Have

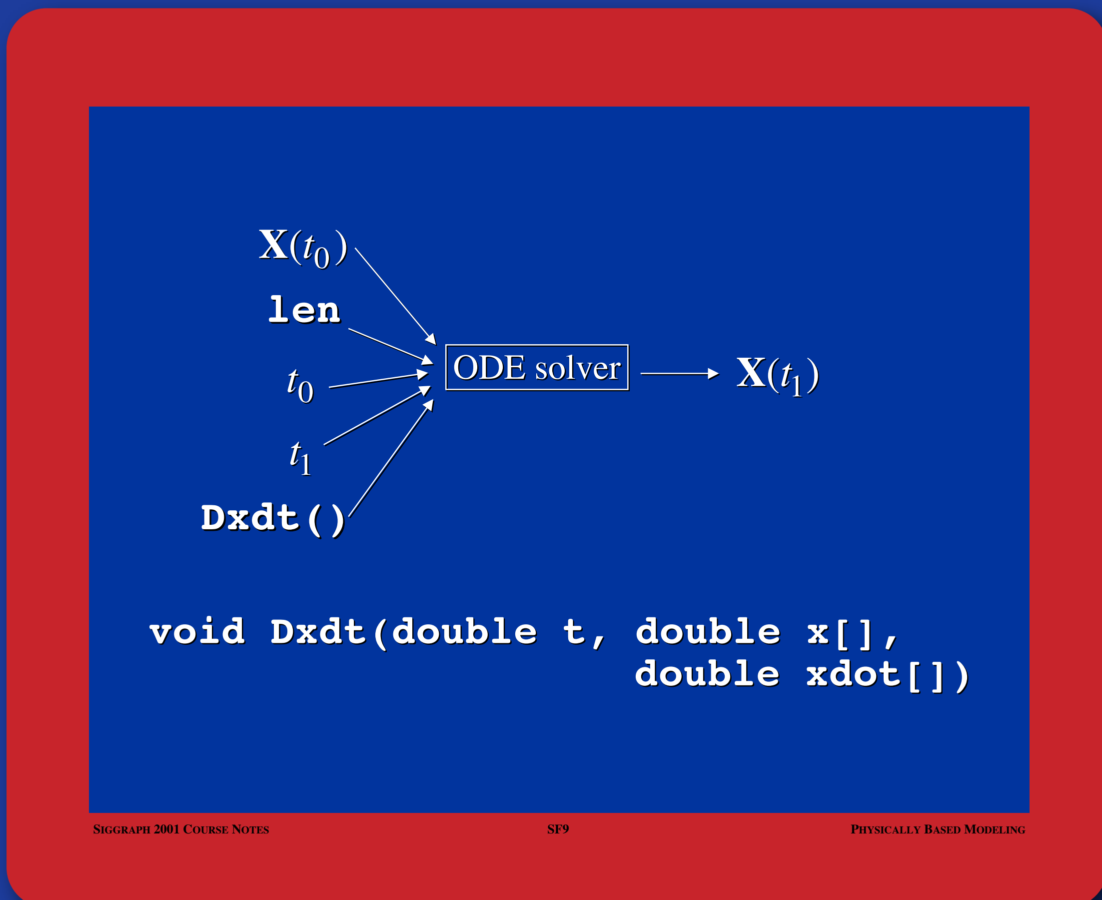


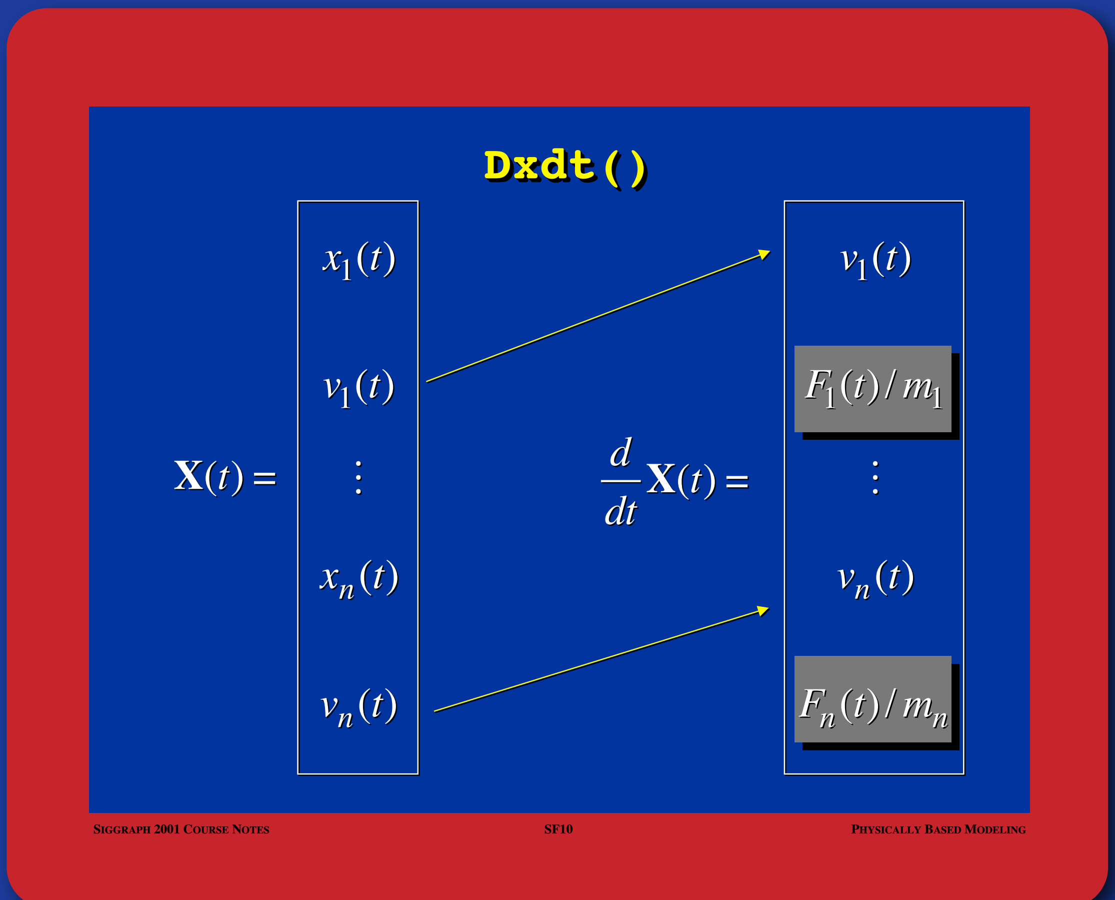
X(t0)
len
t0
t1
Dxdt()
ODE solver
X(t1)
void Dxdt(double t, double x[],
double xdot[])
SIGGRAPH 2001 COURSE NOTES
SF9
PHYSICALLY BASED MODELING



Dxdt()
X(t) =
x1(t)
v1(t)
xn(t)
vn(t)
d/dt X(t) =
v1(t)
F1(t)/m1
vn(t)
Fn(t)/mn
SIGGRAPH 2001 COURSE NOTES
SF10
PHYSICALLY BASED MODELING

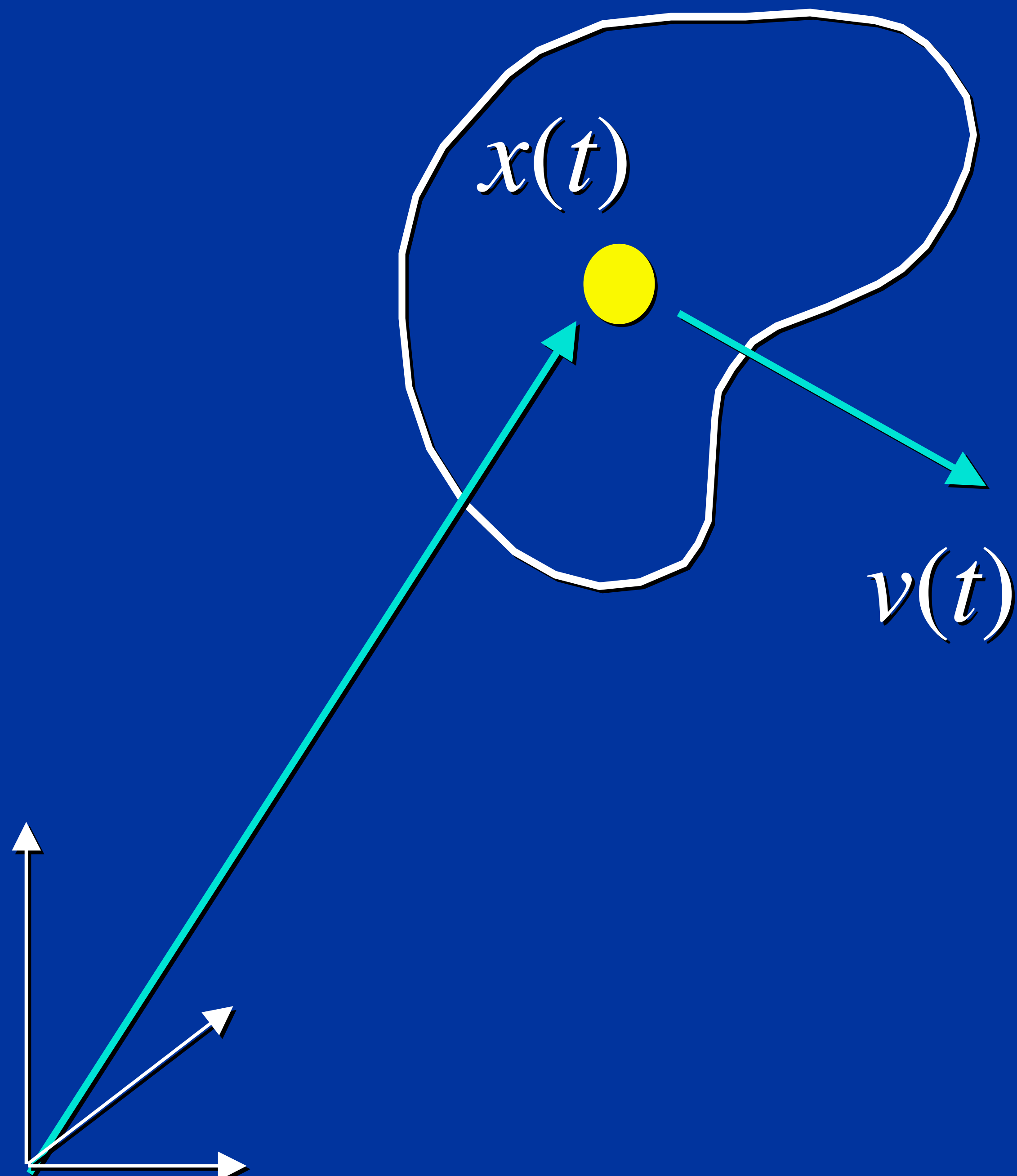
Our Goal





Replicate this approach for rigid bodies.

Rigid Body State



$$\mathbf{X}(t) = \begin{pmatrix} x(t) \\ ? \\ v(t) \\ ? \end{pmatrix}$$

Rigid Body Equation of Motion

$$\frac{d}{dt}\mathbf{X}(t) = \frac{d}{dt}\begin{pmatrix} x(t) \\ ? \\ Mv(t) \\ ? \end{pmatrix} = \begin{pmatrix} ? \end{pmatrix}$$

- Use Momentum P=Mv instead of just v.
- What is this?

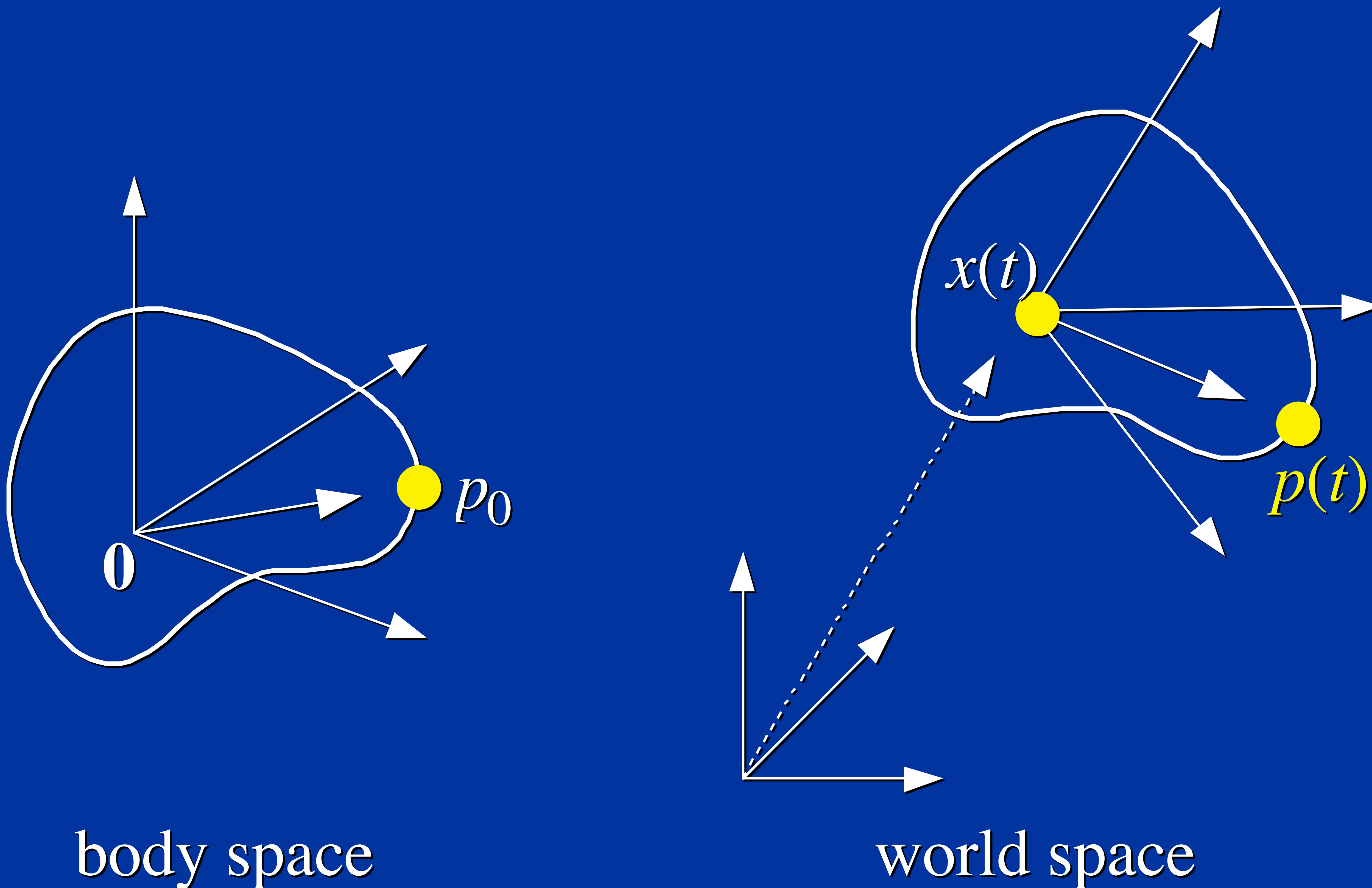
Orientation

We represent orientation as a rotation matrix† $\mathbf{R}(t)$. Points are transformed from body-space to world-space as:

$$p(t) = \mathbf{R}(t)\,p_0 + x(t)$$

---

†He's lying. Actually, we use quaternions.



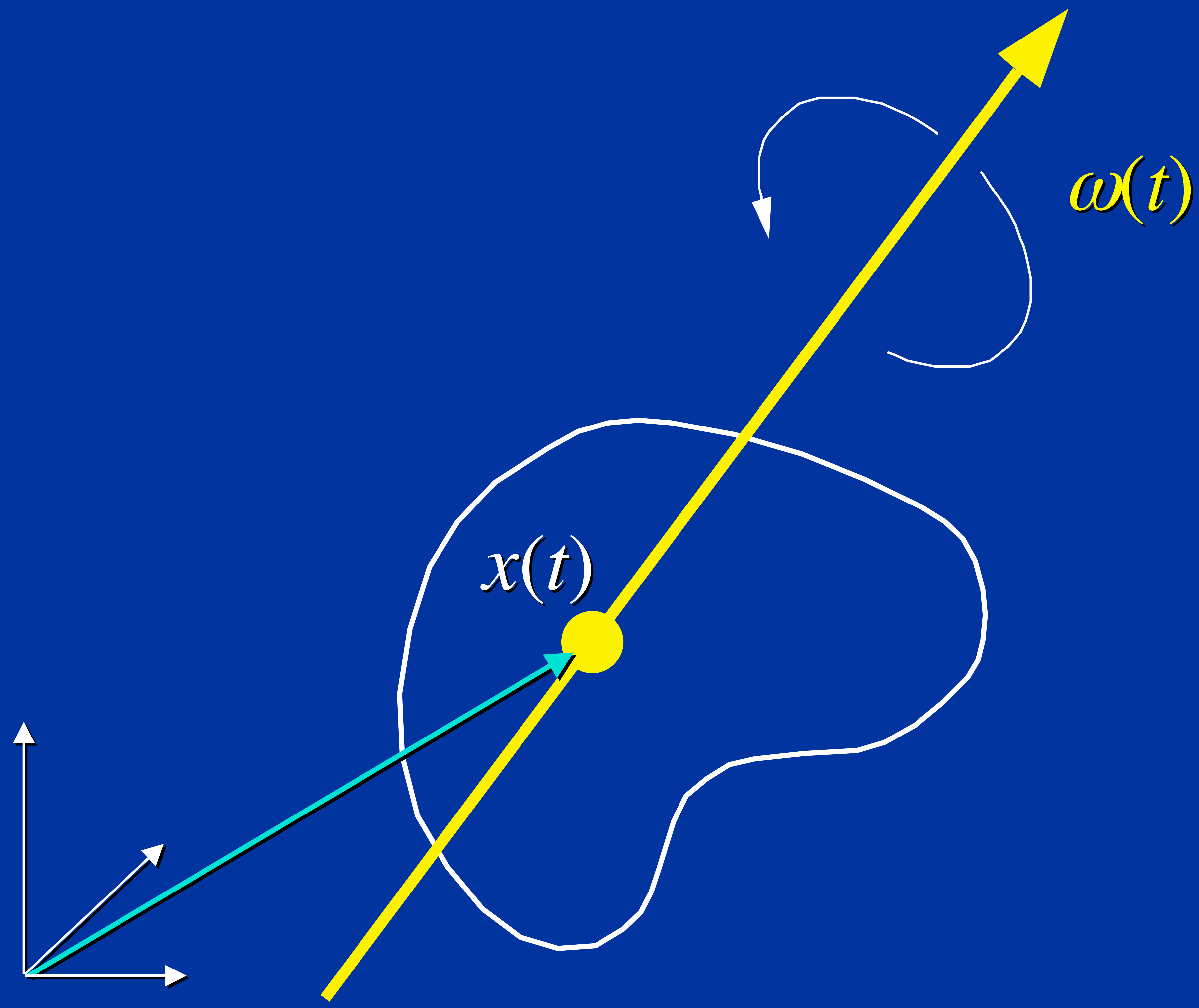
p_0
0
$x(t)$
$p(t)$
body space
world space

Rigid Body Equation of Motion

$$\frac{d}{dt}\mathbf{X}(t) = \frac{d}{dt}\begin{pmatrix} x(t) \\ R(t) \\ Mv(t) \\ ? \end{pmatrix} = \begin{pmatrix} ? \end{pmatrix}$$

- **What is this?**

Angular Velocity Definition

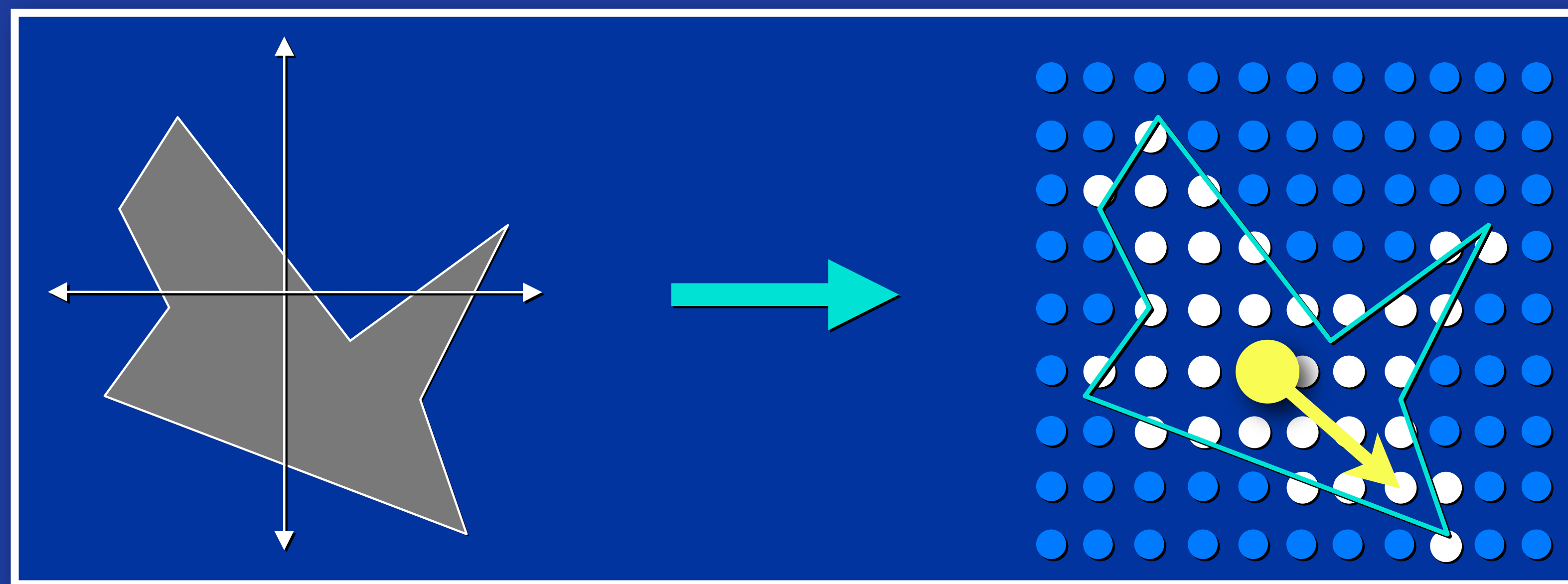


Rigid Body Equation of Motion

$$\frac{d}{dt}\mathbf{X}(t) = \frac{d}{dt}\begin{pmatrix} x(t) \\ R(t) \\ Mv(t) \\ \omega(t) \end{pmatrix} = \begin{pmatrix} ? \end{pmatrix}$$

- **What is this?**

Discretized View

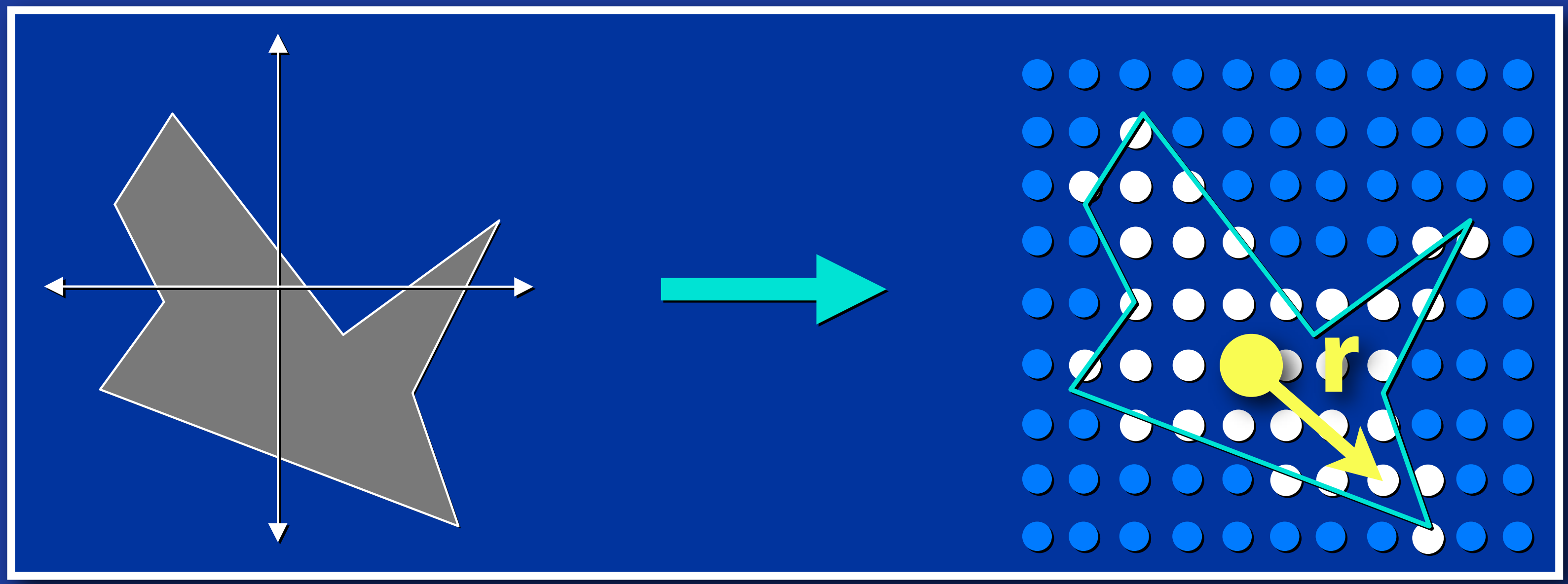


- **Total Mass:** $$M = \sum_i m_i$$

- **Center of Mass:** $$\bar{\mathbf{x}} = \frac{1}{M}\sum_i m_i \mathbf{x}_i$$

- **Relative Position:** $\mathbf{r}_i = \mathbf{x}_i - \bar{\mathbf{x}}$

Discretized View



- **Basic Principles:**
  - **Conservation of Linear Momentum**

$$\frac{d}{dt}\sum_i m_i \dot{\mathbf{x}}_i = 0$$

  - **Conservation of Angular Momentum**

$$\frac{d}{dt}\sum_i m_i \mathbf{r}_i \times \dot{\mathbf{x}}_i = 0$$

Conservation and Forces

Linear Momentum

$$\frac{d}{dt}\sum_i m_i \dot{\mathbf{x}}_i = \sum_i \mathbf{f}_i$$

||(def)

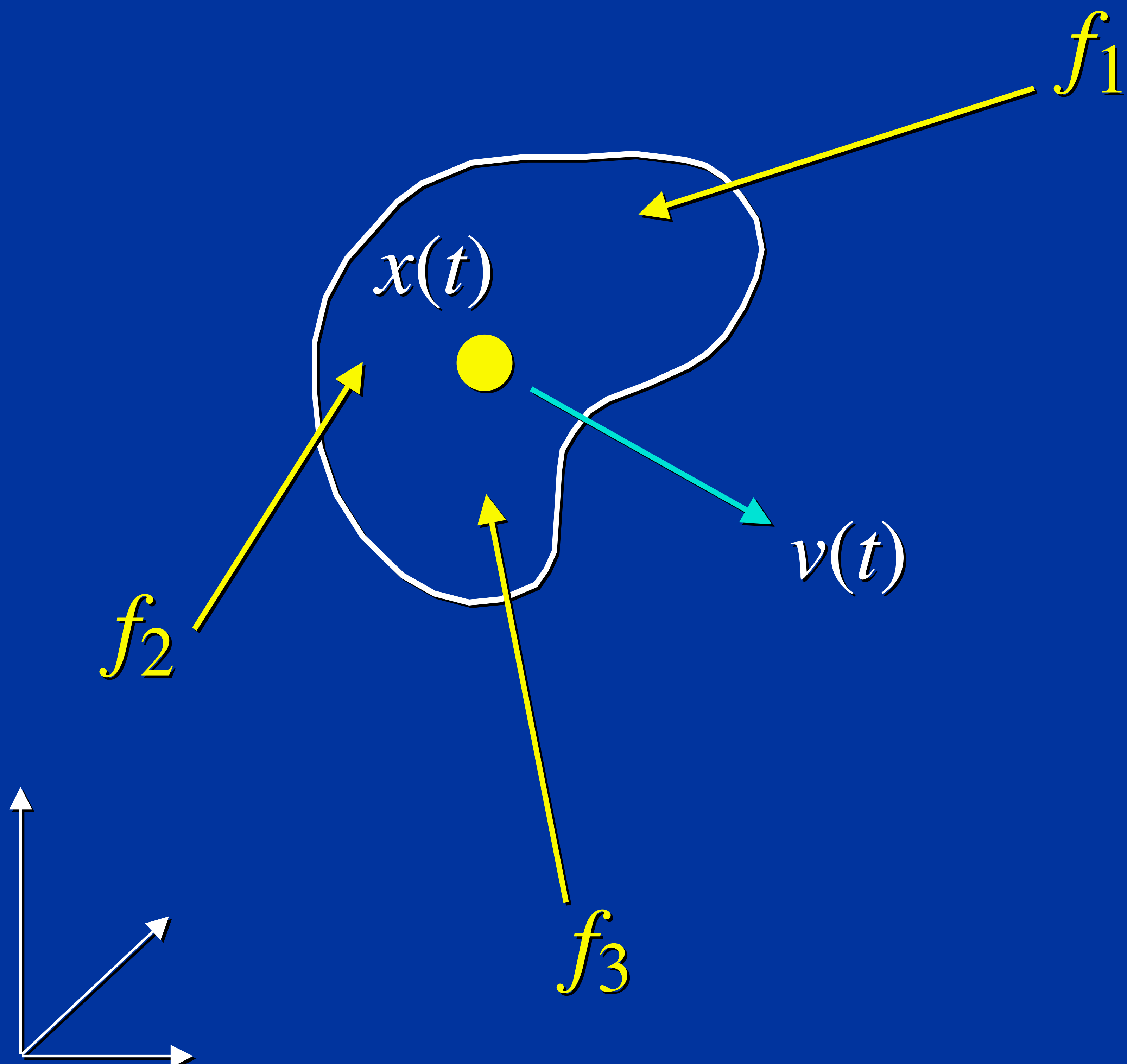
$$\sum_i m_i \ddot{\mathbf{x}}_i = \mathbf{F}$$

$$\left(\bar{\mathbf{x}} = \frac{1}{M}\sum_i m_i \mathbf{x}_i\right)$$

$$\left(M\ddot{\bar{\mathbf{x}}} = \sum_i m_i \ddot{\mathbf{x}}_i\right)$$

$$M\ddot{\bar{\mathbf{x}}} = \mathbf{F}$$

Net Force



$$F(t) = \sum f_i$$

Rigid Body Equation of Motion

$$\frac{d}{dt}\mathbf{X}(t) = \frac{d}{dt}\begin{pmatrix} x(t) \\ R(t) \\ Mv(t) \\ \omega(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ ? \\ F(t) \\ ? \end{pmatrix}$$

- What are these?

Angular Velocity

We represent angular velocity as a vector $\omega(t)$, which encodes both the axis of the spin and the speed of the spin.

How are $\mathbf{R}(t)$ and $\omega(t)$ related?

Angular Velocity

- $\dot{\mathbf{R}}(t)$ and $\omega(t)$ are related by:

$$\frac{d}{dt}\mathbf{R}(t) = \begin{pmatrix} 0 & -\omega_z(t) & \omega_y(t) \\ \omega_z(t) & 0 & -\omega_x(t) \\ -\omega_y(t) & \omega_x(t) & 0 \end{pmatrix} \mathbf{R}(t)$$

$$= \omega(t)^* \mathbf{R}(t)$$

w^* **can be viewed as the matrix form of** $-(w\times)$

Rigid Body Equation of Motion

$$\frac{d}{dt}\mathbf{X}(t) = \frac{d}{dt}\begin{pmatrix} x(t) \\ \mathbf{R}(t) \\ Mv(t) \\ \langle \omega(t) \rangle \end{pmatrix} = \begin{pmatrix} v(t) \\ \omega(t)^{*}\mathbf{R}(t) \\ F(t) \\ ? \end{pmatrix}$$

Need to relate $\dot{\omega}(t)$ and mass distribution to $F(t)$.

Conservation and Forces

Linear Momentum

$$\frac{d}{dt}\sum_i m_i \dot{\mathbf{x}}_i = \sum_i \mathbf{f}_i$$

$$\| \text{(def)}$$

$$\sum_i m_i \ddot{\mathbf{x}}_i = \mathbf{F}$$

$$\left(\bar{\mathbf{x}} = \frac{1}{M}\sum_i m_i \mathbf{x}_i \right)$$

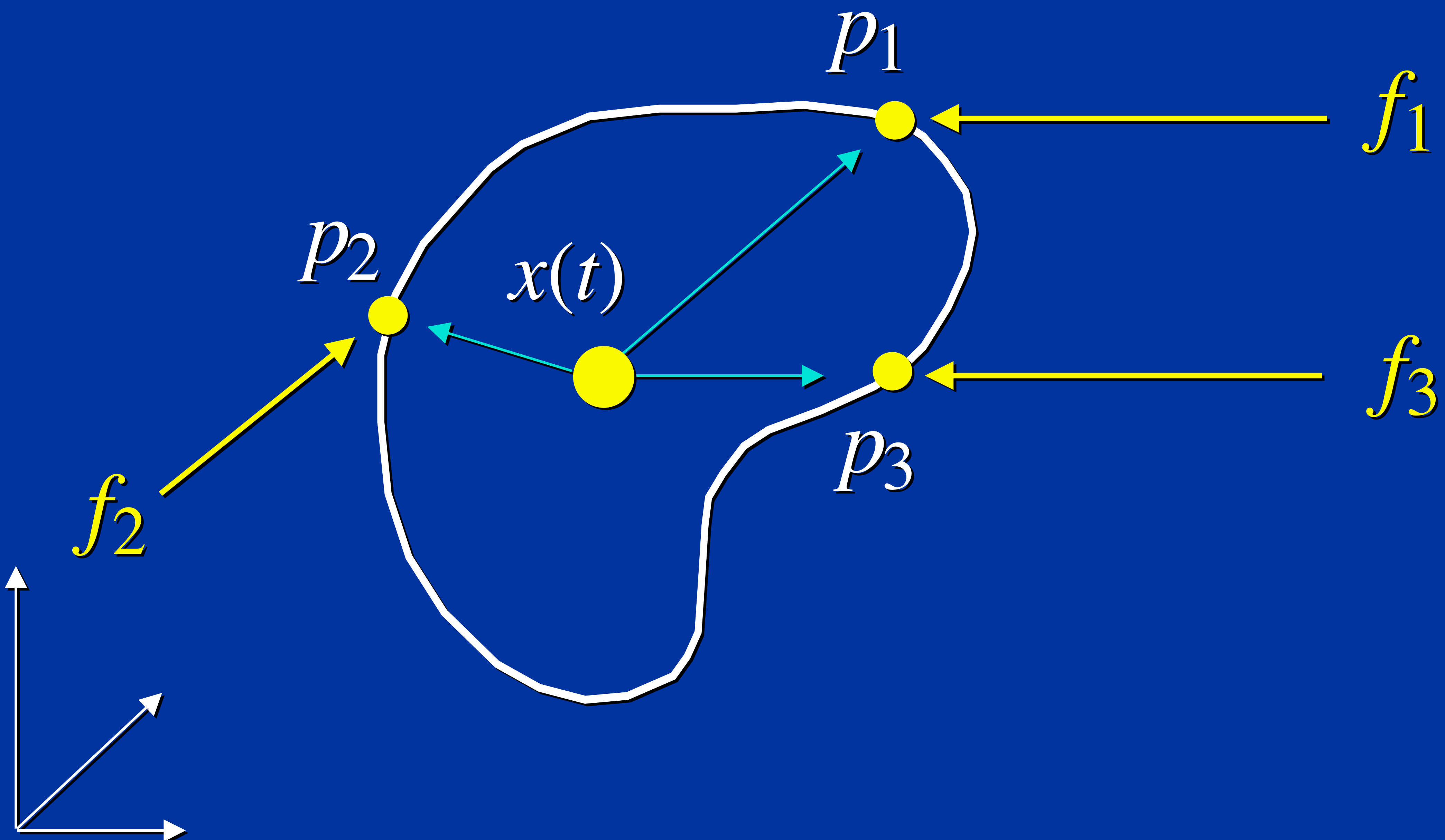
$$\left(M\ddot{\bar{\mathbf{x}}} = \sum_i m_i \ddot{\mathbf{x}}_i \right)$$

$$M\ddot{\bar{\mathbf{x}}} = \mathbf{F}$$

Angular Momentum

$$\frac{d}{dt}\sum_i m_i \mathbf{r}_i \times \dot{\mathbf{x}}_i = \sum_i \mathbf{r}_i \times \mathbf{f}_i$$

Net Torque



$$\tau(t) = \sum \Big(p_i - x(t) \Big) \times f_i$$

Conservation and Forces

Linear Momentum

$$\frac{d}{dt}\sum_i m_i\dot{\mathbf{x}}_i = \sum_i \mathbf{f}_i$$

|| (def)

$$\sum_i m_i\ddot{\mathbf{x}}_i = \mathbf{F}$$

$$\left(\bar{\mathbf{x}} = \frac{1}{M}\sum_i m_i\mathbf{x}_i\right)$$

$$\left(M\ddot{\bar{\mathbf{x}}} = \sum_i m_i\ddot{\mathbf{x}}_i\right)$$

$$M\ddot{\bar{\mathbf{x}}} = \mathbf{F}$$

Angular Momentum

$$\frac{d}{dt}\sum_i m_i\mathbf{r}_i \times \dot{\mathbf{x}}_i = \sum_i \mathbf{r}_i \times \mathbf{f}_i$$

$$\frac{d}{dt}\sum_i m_i\mathbf{r}_i \times \dot{\mathbf{x}}_i = \tau$$

$$\frac{d}{dt}\sum_i m_i\mathbf{r}_i \times \omega \times \mathbf{r}_i = \tau$$

$$\frac{d}{dt}\sum_i m_i\mathbf{r}_i^*\mathbf{r}_i^*\omega = \tau$$

Discrete Inertia

$$I = \sum_i m_i \mathbf{r}_i^* \mathbf{r}_i^*$$

$$I = \sum_i \left(m_i \begin{bmatrix} -y^2 - z^2 & xy & xz \\ xy & -x^2 - z^2 & yz \\ xz & yz & -x^2 - y^2 \end{bmatrix} \right)$$

Conservation and Forces

Linear Momentum

$$\frac{d}{dt}\sum_i m_i \dot{\mathbf{x}}_i = \sum_i \mathbf{f}_i$$

$||$ (def)

$$\sum_i m_i \ddot{\mathbf{x}}_i = \mathbf{F}$$

$$\left(\bar{\mathbf{x}} = \frac{1}{M}\sum_i m_i \mathbf{x}_i\right)$$

$$\left(M\ddot{\bar{\mathbf{x}}} = \sum_i m_i \ddot{\mathbf{x}}_i\right)$$

$$M\ddot{\bar{\mathbf{x}}} = \mathbf{F}$$

Angular Momentum

$$\frac{d}{dt}\sum_i m_i \mathbf{r}_i \times \dot{\mathbf{x}}_i = \sum_i \mathbf{r}_i \times \mathbf{f}_i$$

$$\frac{d}{dt}\sum_i m_i \mathbf{r}_i \times \dot{\mathbf{x}}_i = \tau$$

$$\frac{d}{dt}\sum_i m_i \mathbf{r}_i \times \omega \times \mathbf{r}_i = \tau$$

$$\frac{d}{dt}\sum_i m_i \mathbf{r}_i^* \mathbf{r}_i^* \omega = \tau$$

$$\frac{d}{dt} I\omega = \tau$$

Rigid Body Equation of Motion

$$\frac{d}{dt}\mathbf{X}(t) = \frac{d}{dt}\begin{pmatrix} x(t) \\ \mathbf{R}(t) \\ Mv(t) \\ \mathbf{I}(t)\omega(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ \omega(t)^{*}\mathbf{R}(t) \\ F(t) \\ \tau(t) \end{pmatrix}$$

$P(t)$ – linear momentum

$L(t)$ – angular momentum

Discrete Inertia

$$I = \sum_i m_i \mathbf{r}_i^* \mathbf{r}_i^*$$

$$I = \sum_i \left(m_i \begin{bmatrix} -y^2 - z^2 & xy & xz \\ xy & -x^2 - z^2 & yz \\ xz & yz & -x^2 - y^2 \end{bmatrix} \right)$$

Continuous Inertia

$$\mathbf{I}(t) = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

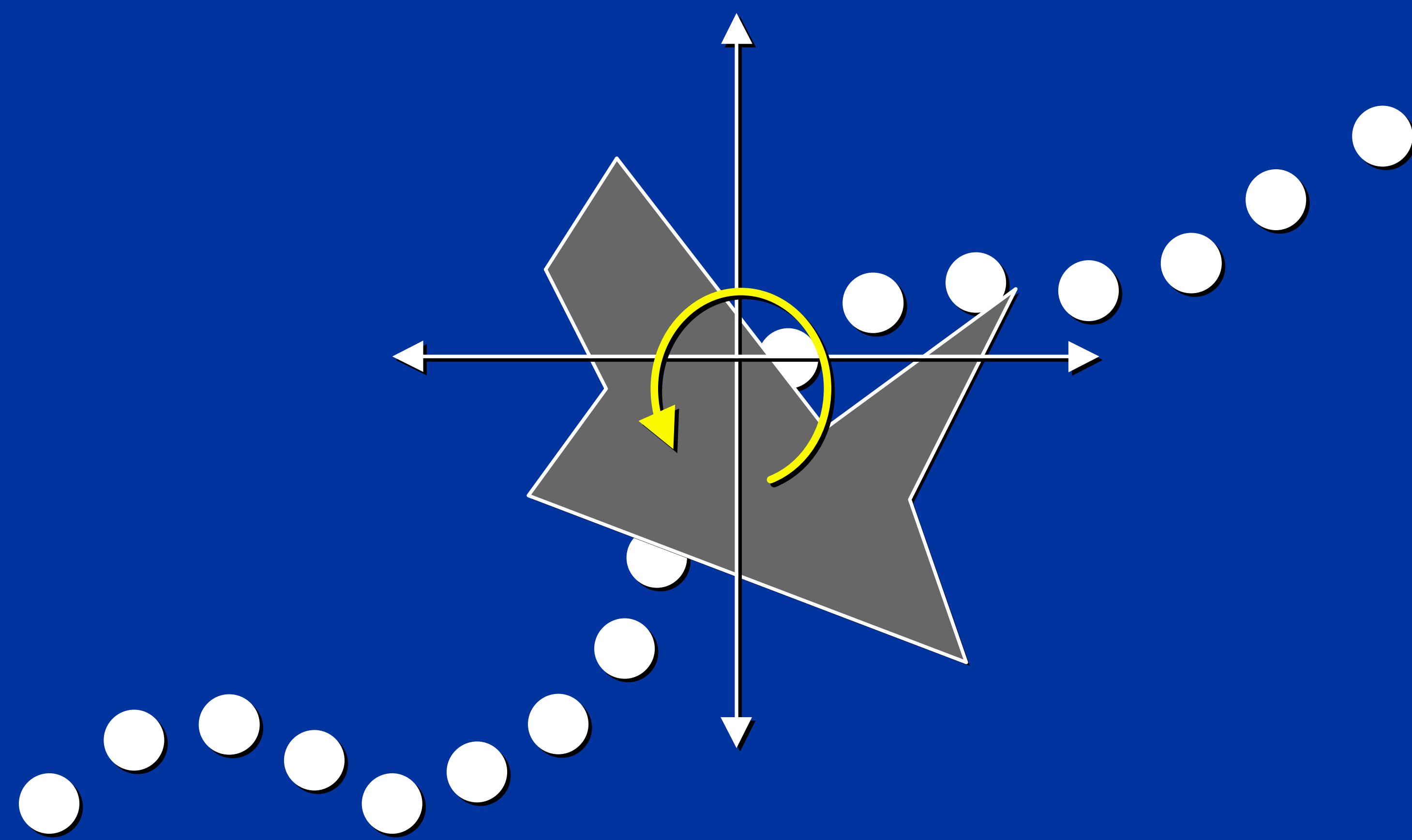
diagonal terms

$$I_{xx} = M \int_V (y^2 + z^2)\, dV$$

off-diagonal terms

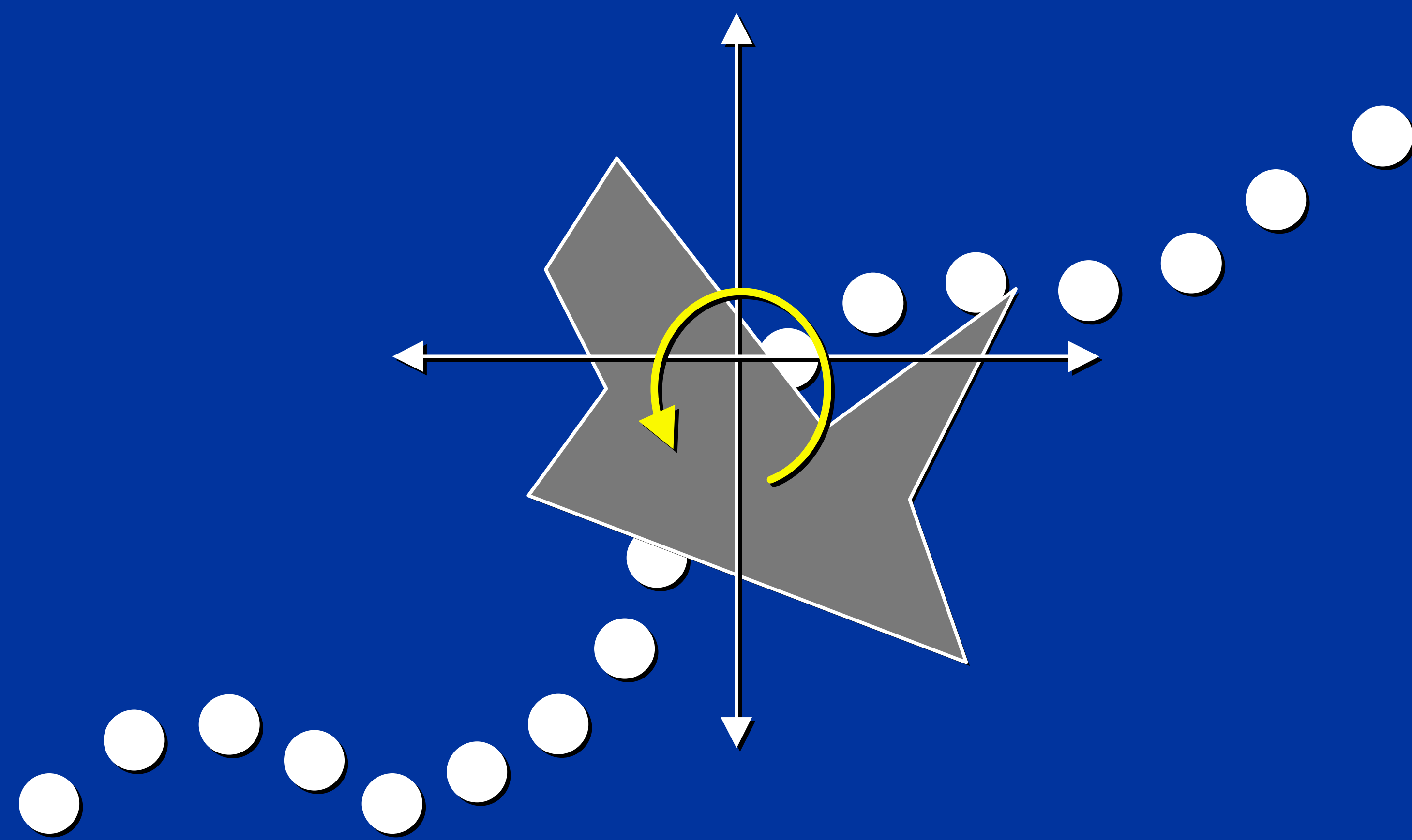
$$I_{xy} = -M \int_V xy\, dV$$

Inertia Tensors Vary in World Space...



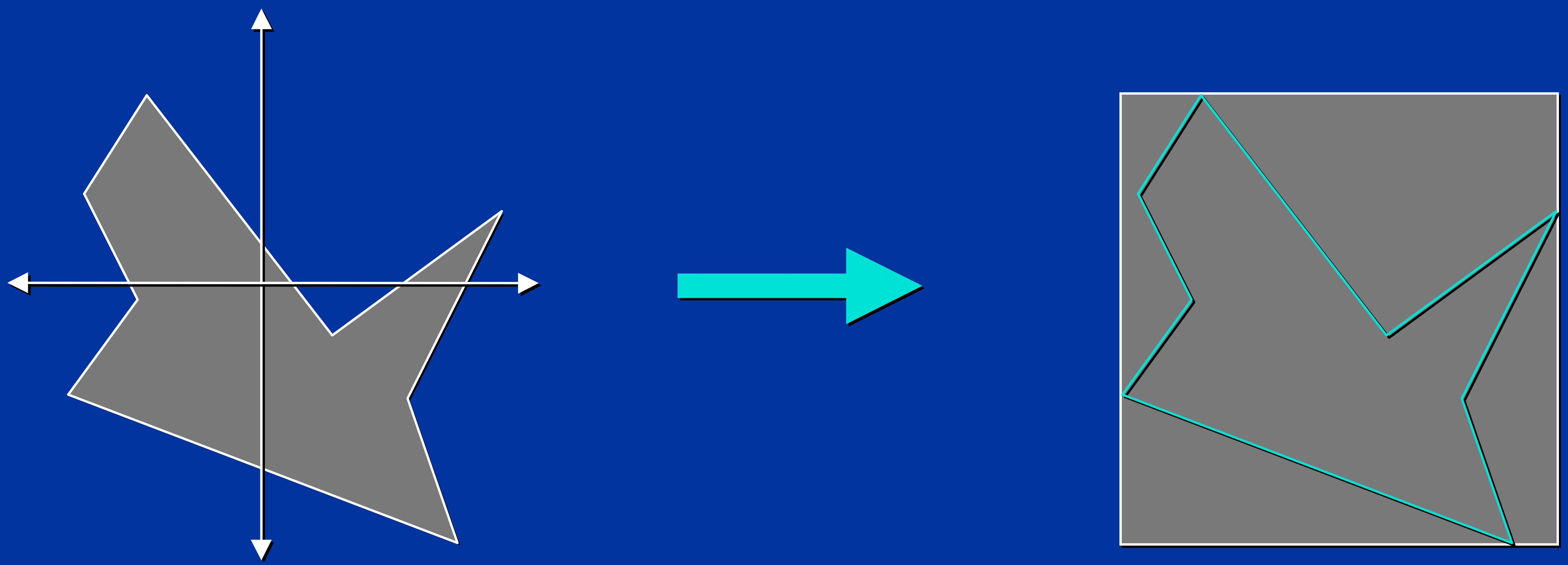
$$I_{xx} = M \int_V (y^2 + z^2)\, dV \qquad I_{xy} = -M \int_V xy\, dV$$

... but are Constant in Body Space



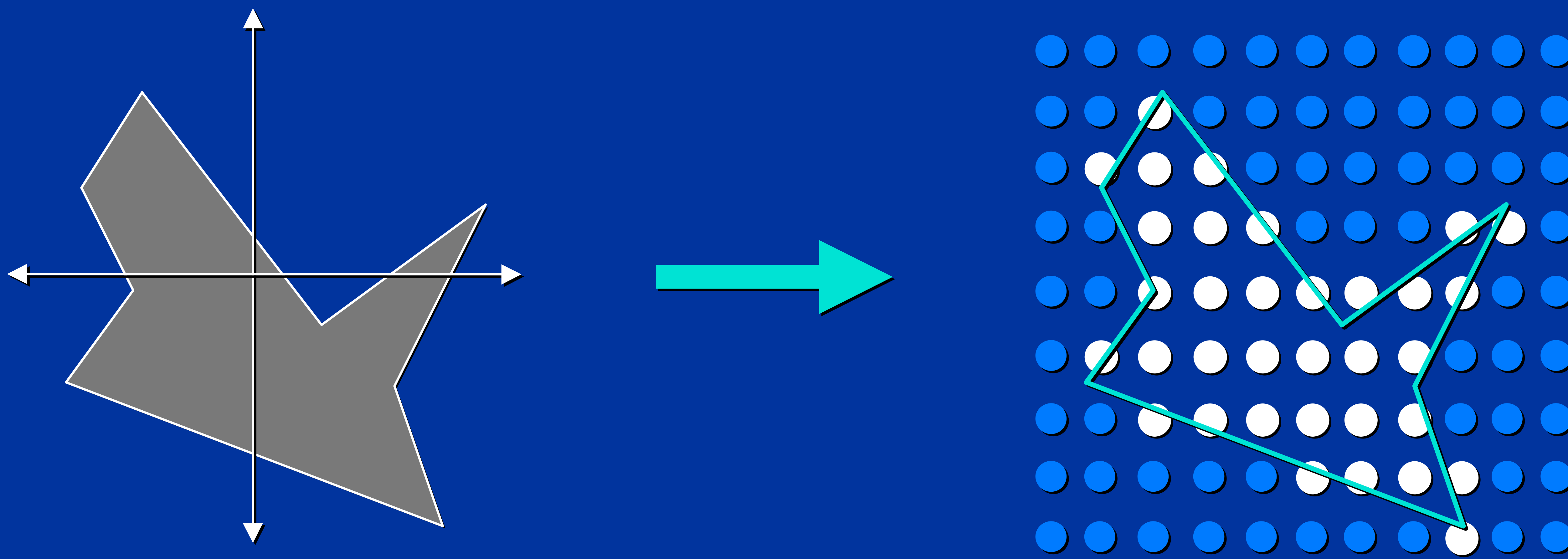
$$\mathbf{I}(t) = \mathbf{R}(t)\mathbf{I}_{\text{body}}\mathbf{R}(t)^T$$

Approximating I_{body}: Bounding Boxes



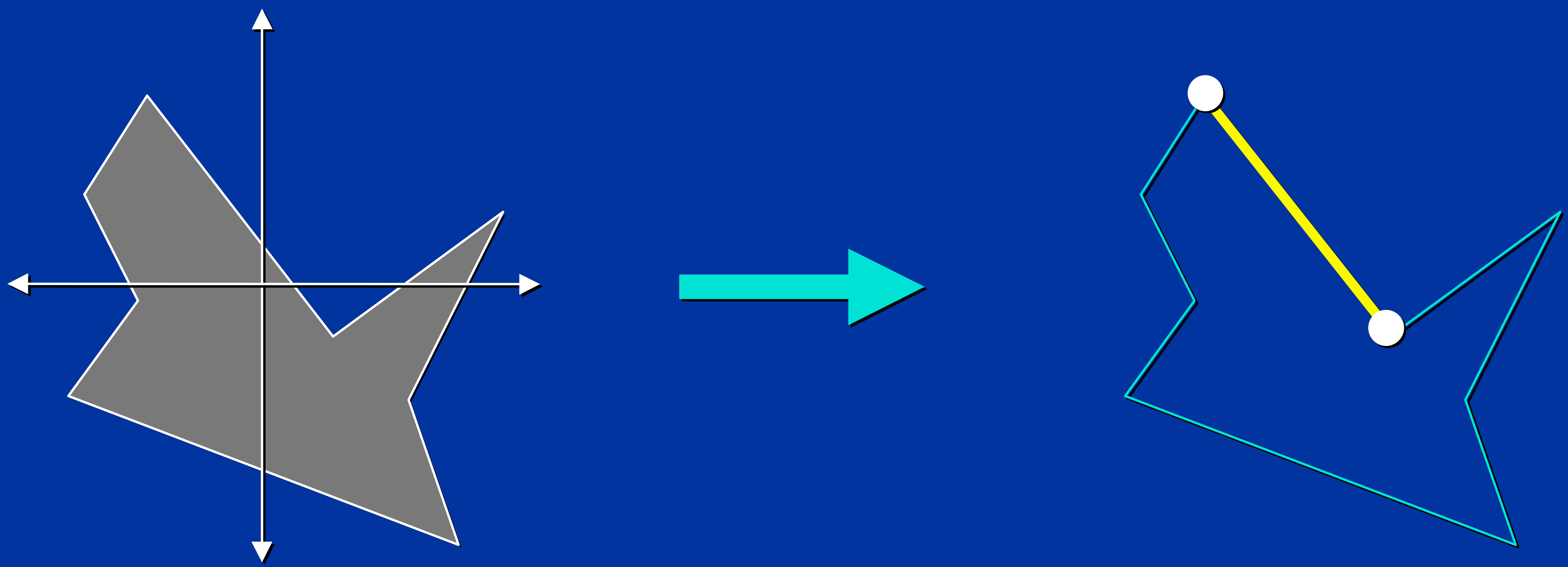
Pros: Simple.
Cons: Bounding box may not be a good fit.
Inaccurate.

Approximating I_{body}: Point Samping



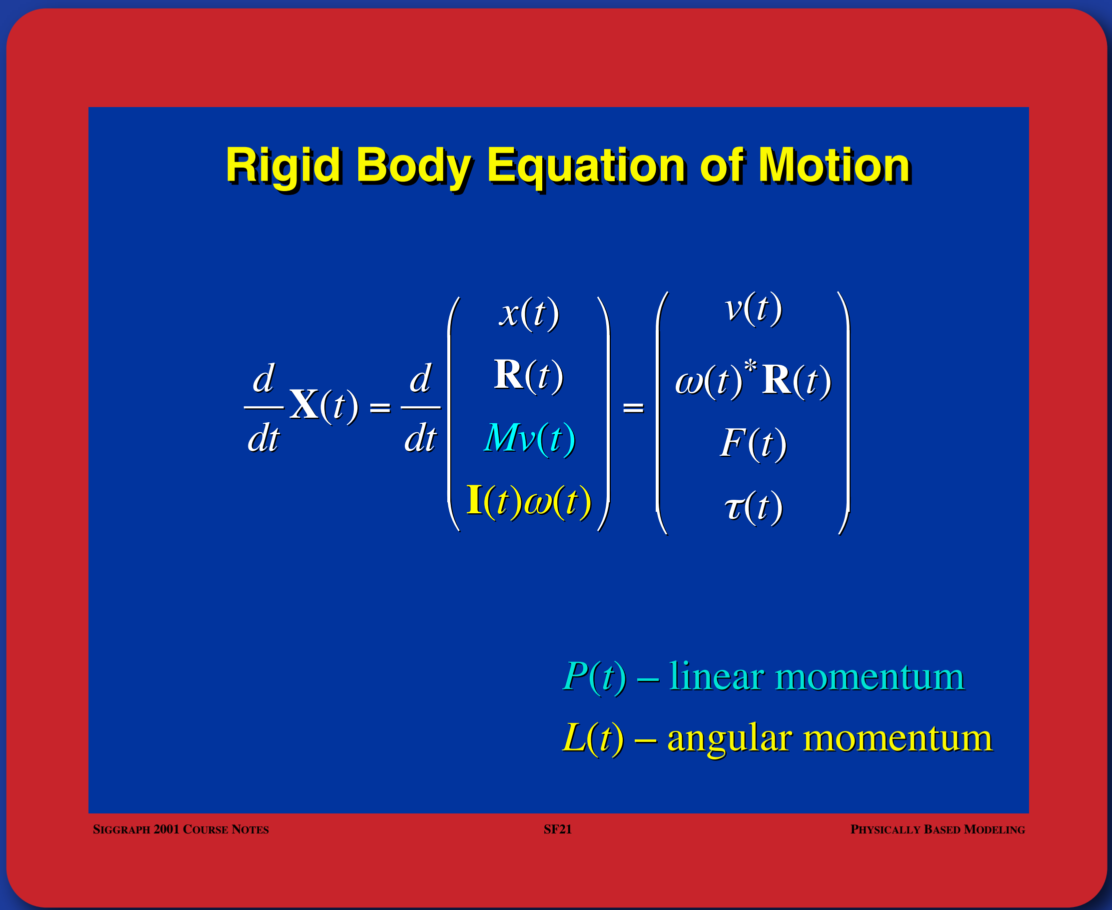
Pros: Simple, fairly accurate, no B-rep needed.
Cons: Expensive, requires volume test.

Computing I_{body}: Green's Theorem (2x!)



Pros: Simple, exact, no volumes needed.
Cons: Requires boundary representation.
Code: http://www.acm.org/jgt/papers/Mirtich96

Summary


Rigid Body Equation of Motion
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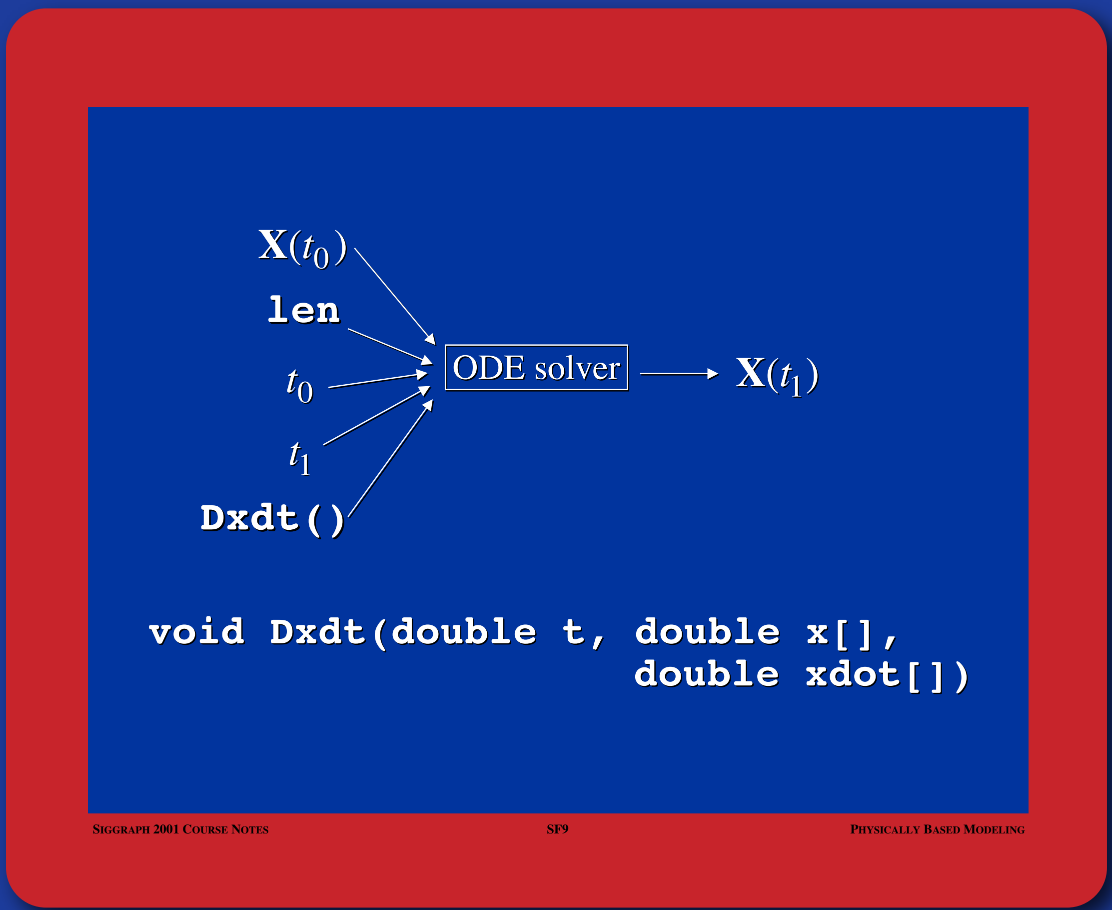
$P(t)$ – linear momentum
$L(t)$ – angular momentum
Siggraph 2001 Course Notes
SF21
Physically Based Modeling


$\mathbf{X}(t_0)$
len
t_0
t_1
Dxdt()
ODE solver
$\mathbf{X}(t_1)$
void Dxdt(double t, double x[],
double xdot[])
Siggraph 2001 Course Notes
SF9
Physically Based Modeling

What's in the Course Notes

1. Implementation of **`Dxdt()`** for rigid bodies (bookkeeping, data structures, computations)
2. Quaternions—derivations and code
3. Miscellaneous formulas and examples
4. Derivations for force and torque equations, center of mass, inertia tensor, rotation equations, velocity/acceleration of points

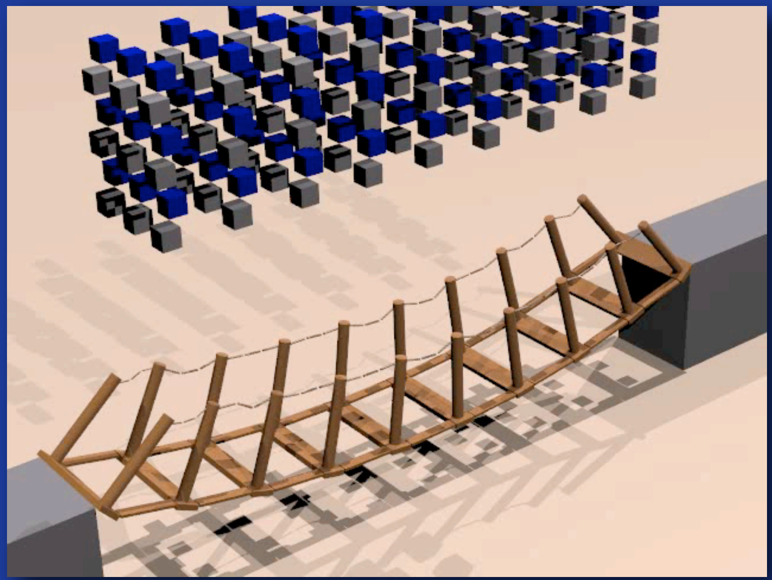
Example

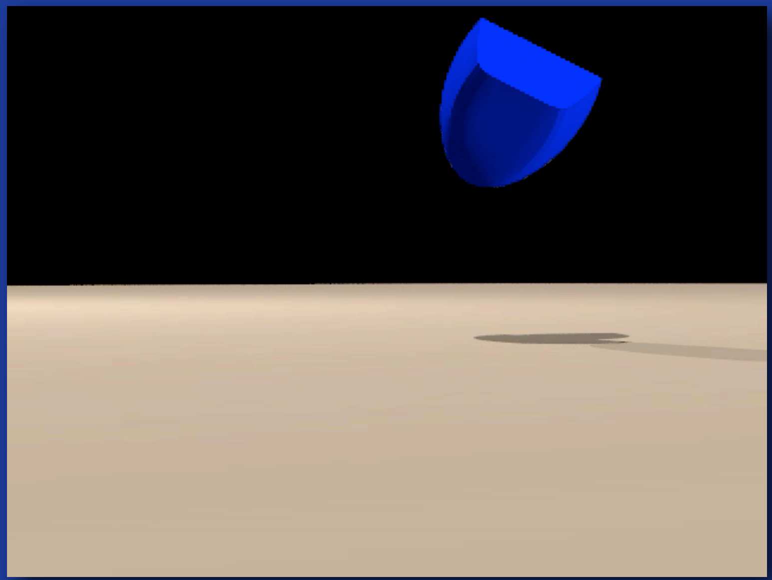
Example

These simulations could never have been created by hand.

Question

- What Kind of Collisions Are Possible?
  - Geometrically?
  - Physically?





- How can these be detected?
- What algorithm can handle them?