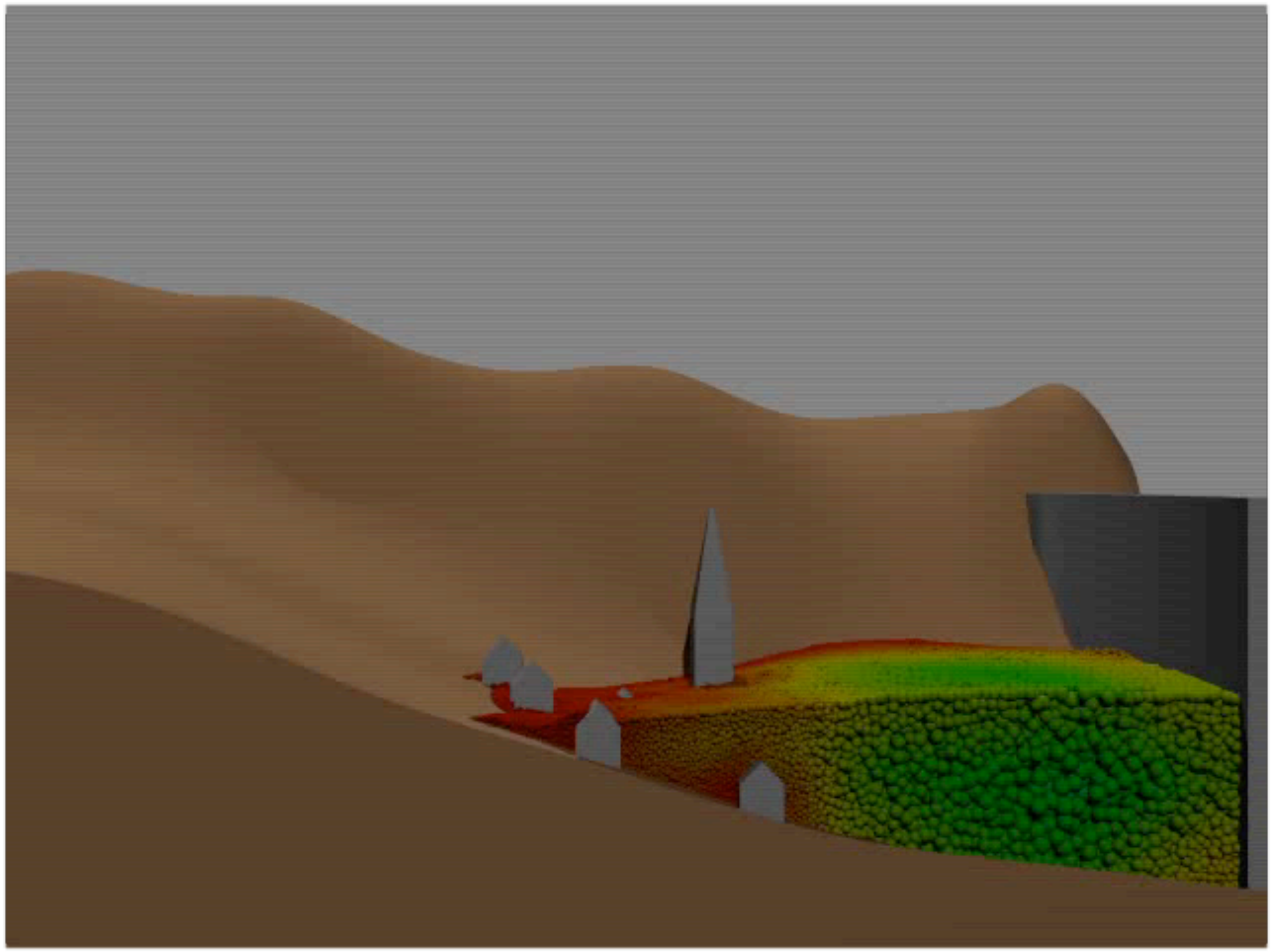
# Particle Fluids



Adrien Treuille

# Overview

- Smoothed Particle Hydrodynamics (SPH) Basics
- Fluid Behavior
- The Navier-Stokes Equations
- SPH Simulation
- Rendering
- Results
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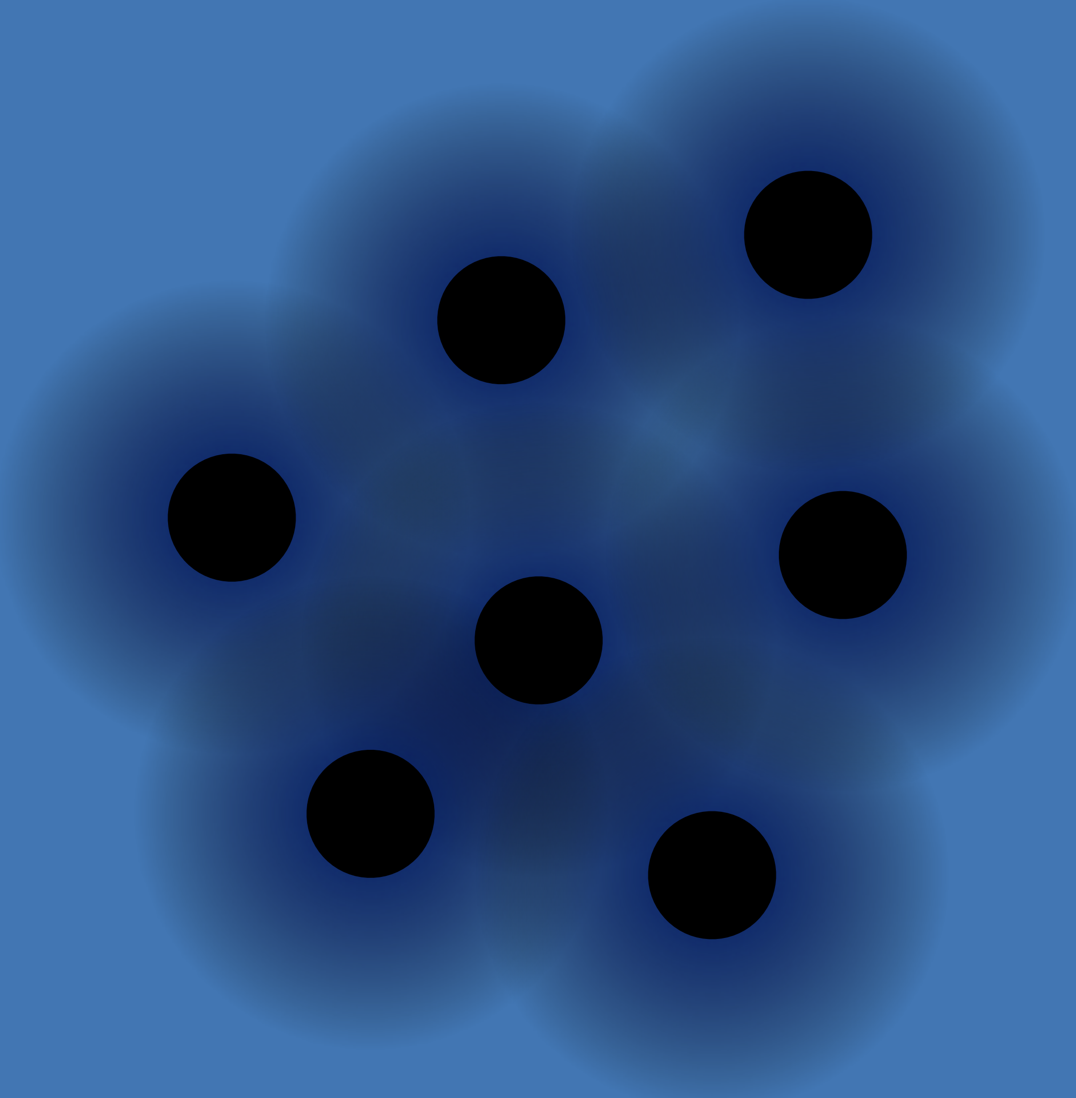
# Particle Model



**Properties**

- Position - $\boldsymbol{x}$
- Velocity - $\boldsymbol{v}$
- Mass - $m$
  - Density - $\rho$

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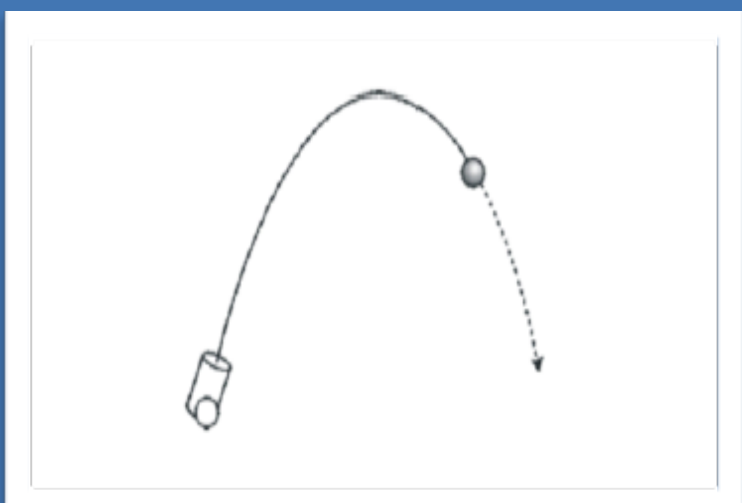
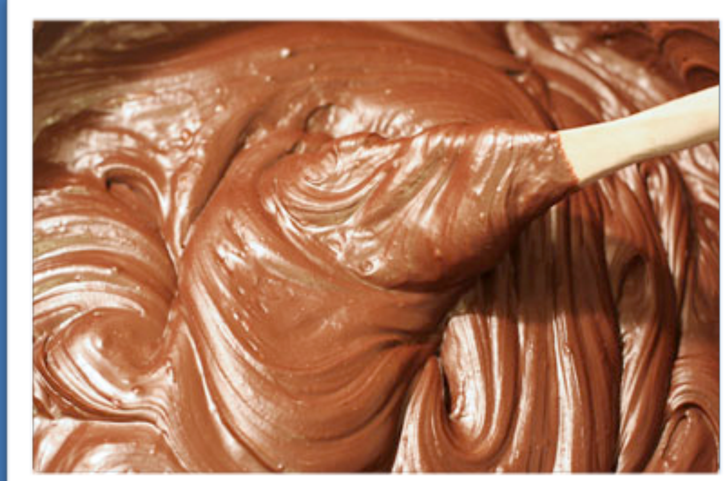
# Fluid Behavior



Gravity

Resistance to Compression



Viscosity

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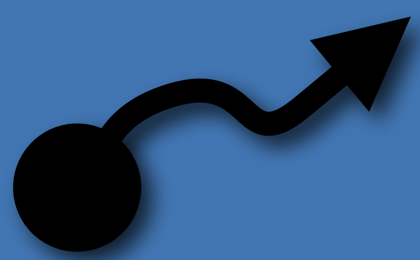
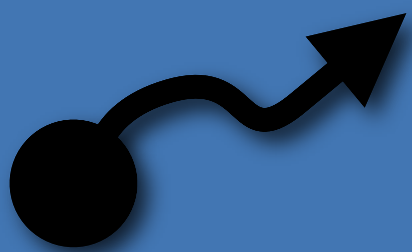
# Fluid Forces



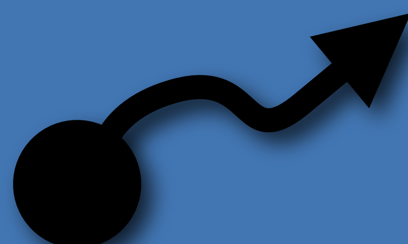
$$m\dot{\mathbf{v}} = \mathbf{f}$$

# Fluid Forces

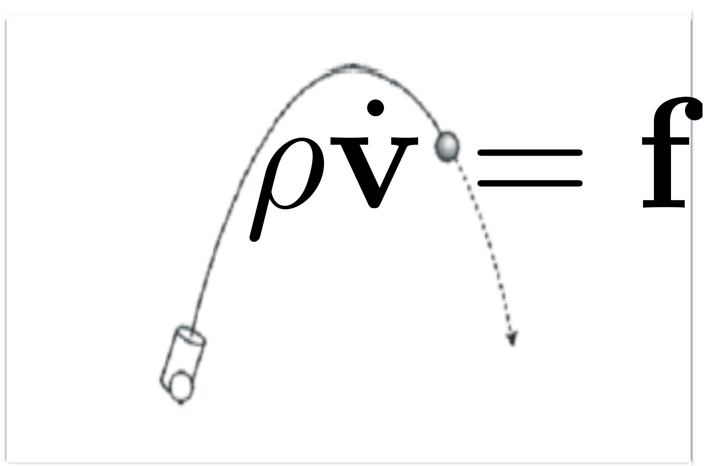


$$\rho \dot{\mathbf{v}} = \mathbf{f}$$

# Fluid Forces



$$\rho \dot{\mathbf{v}} = \mathbf{f}$$

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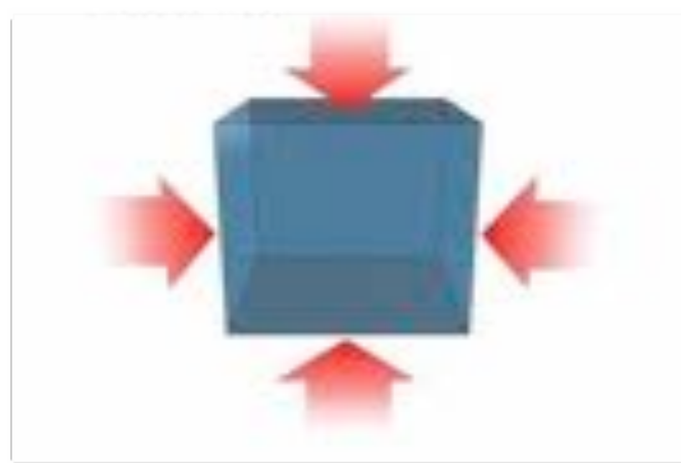

Resistance to Compression

Viscosity

# Fluid Forces

$$\rho\dot{\mathbf{v}} = \mathbf{f}_{\text{gravity}} + \mathbf{f}_{\text{pressure}} + \mathbf{f}_{\text{viscosity}}$$

Gravity

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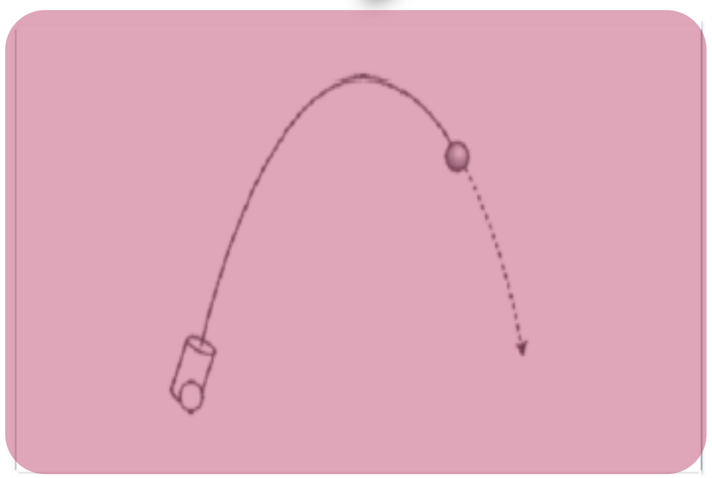
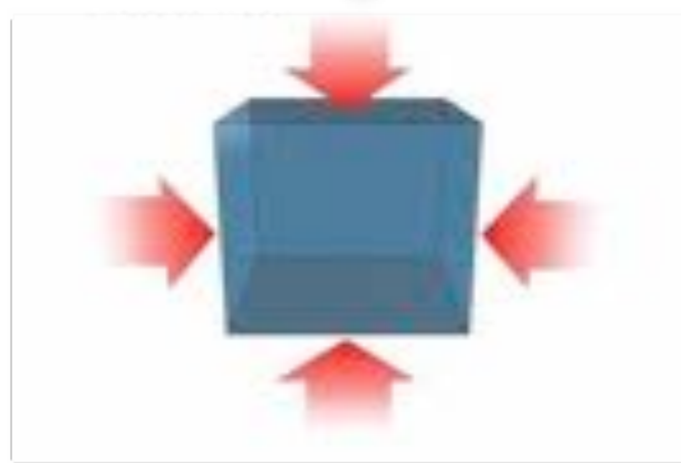
Viscosity

# Fluid Forces

$$\rho\dot{\mathbf{v}} = \mathbf{f}_{\text{gravity}} + \mathbf{f}_{\text{pressure}} + \mathbf{f}_{\text{viscosity}}$$



Gravity



Resistance to Compression

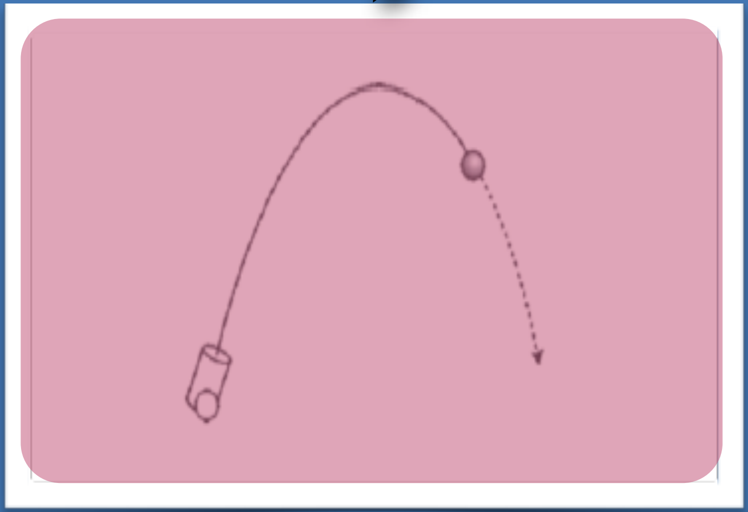
Viscosity

# Gravity

$$\mathbf{f}_{\text{gravity}} = \rho \mathbf{g}$$

# Fluid Forces

$$\rho\dot{\mathbf{v}} = \mathbf{f}_{\text{gravity}} + \mathbf{f}_{\text{pressure}} + \mathbf{f}_{\text{viscosity}}$$



Gravity

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Viscosity

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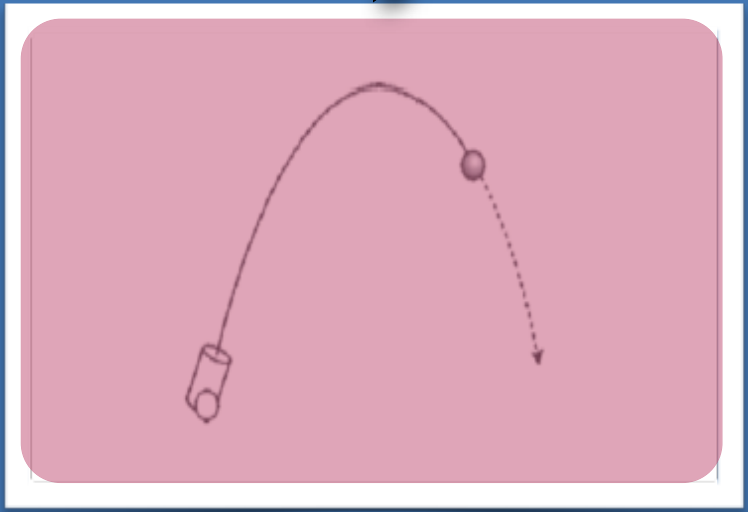

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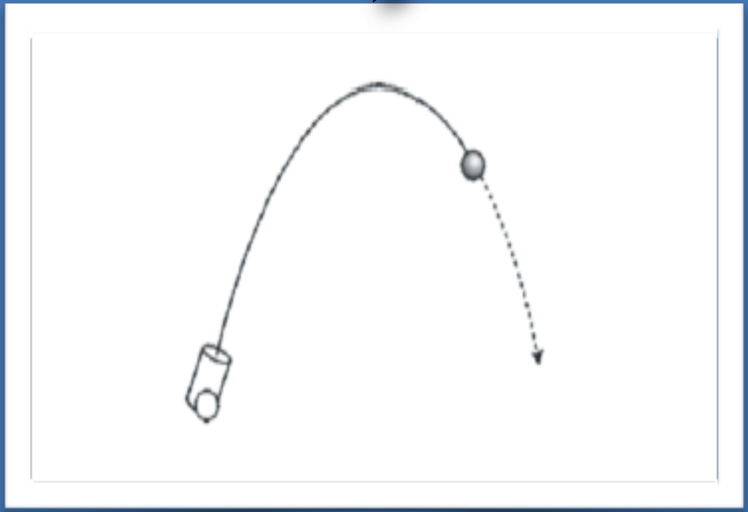

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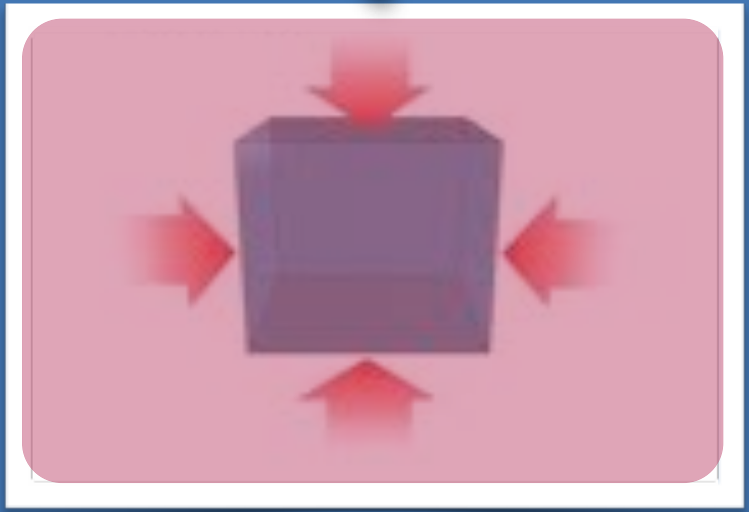
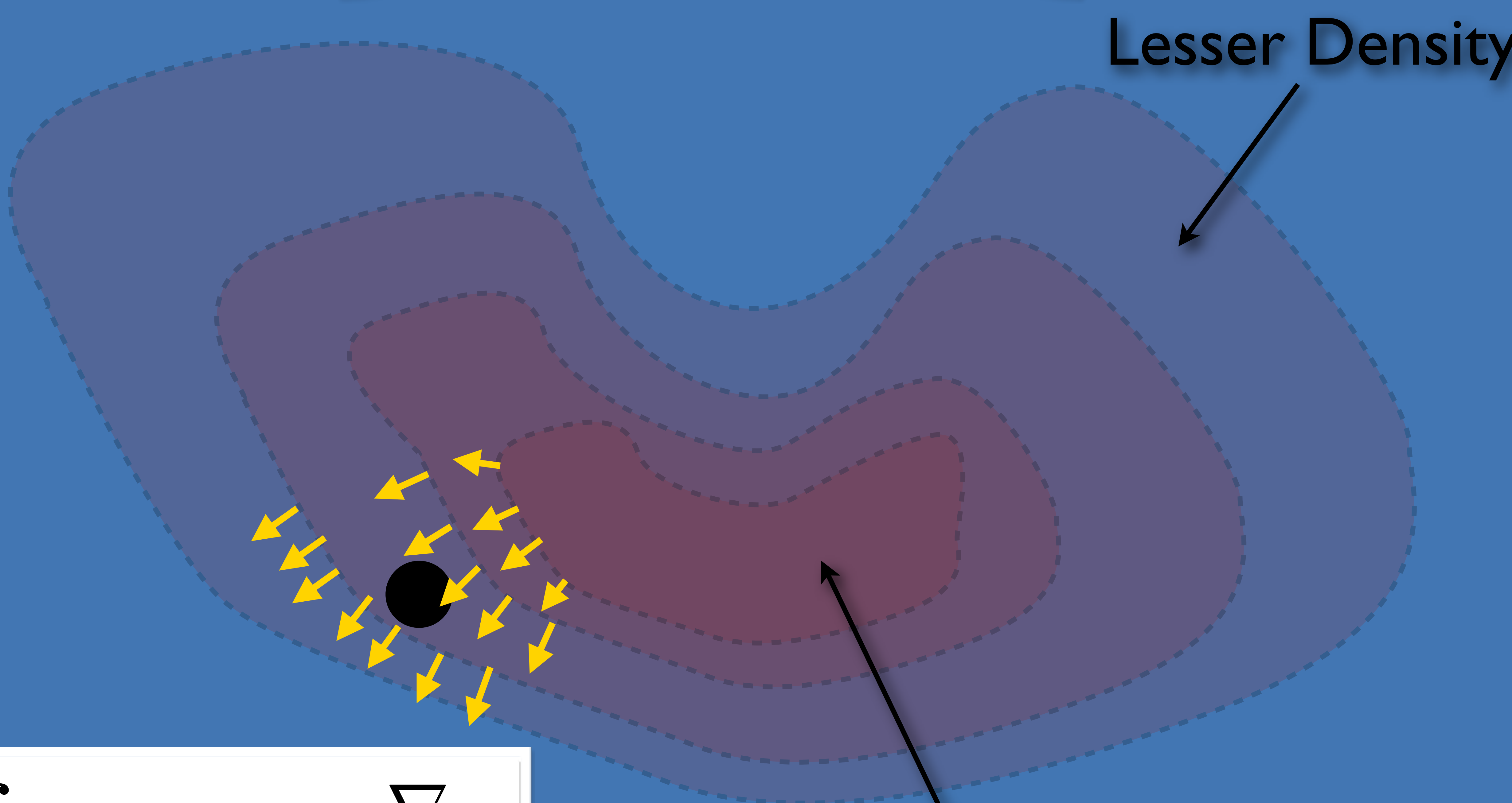


Resistance to Compression

Viscosity

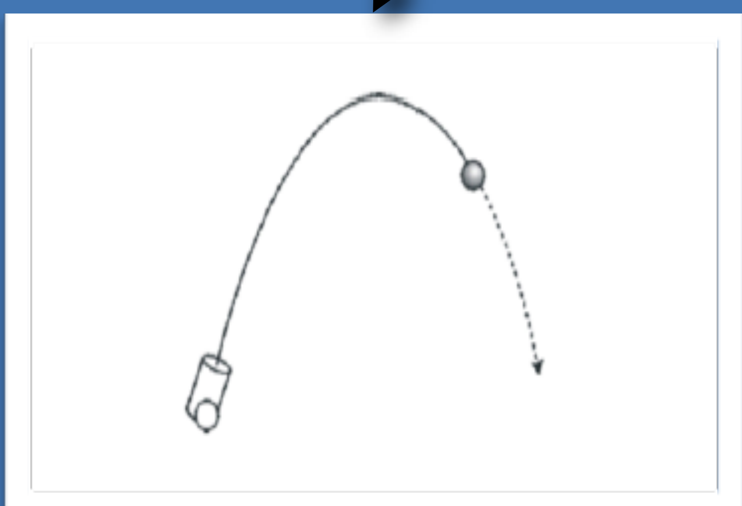
# Pressure




$$\mathbf{f}_{\text{pressure}} = -\nabla p$$
$$p \propto \rho$$

# Fluid Forces

$$\rho\dot{\mathbf{v}} = \rho\mathbf{g} + \mathbf{f}_{\text{pressure}} + \mathbf{f}_{\text{viscosity}}$$



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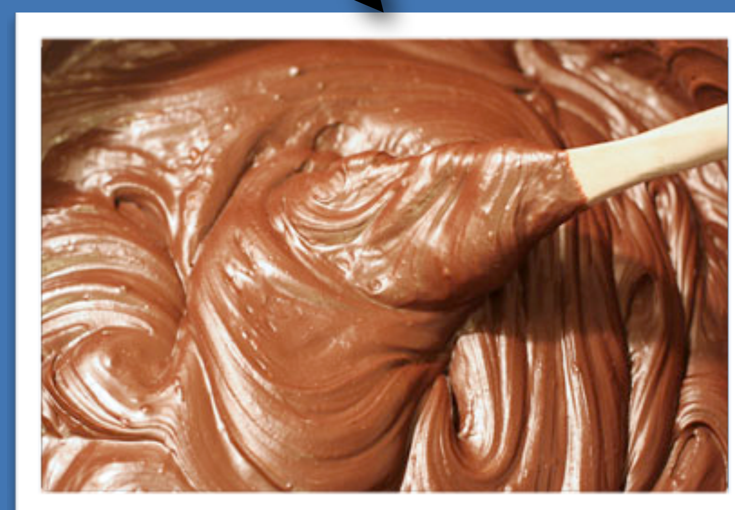
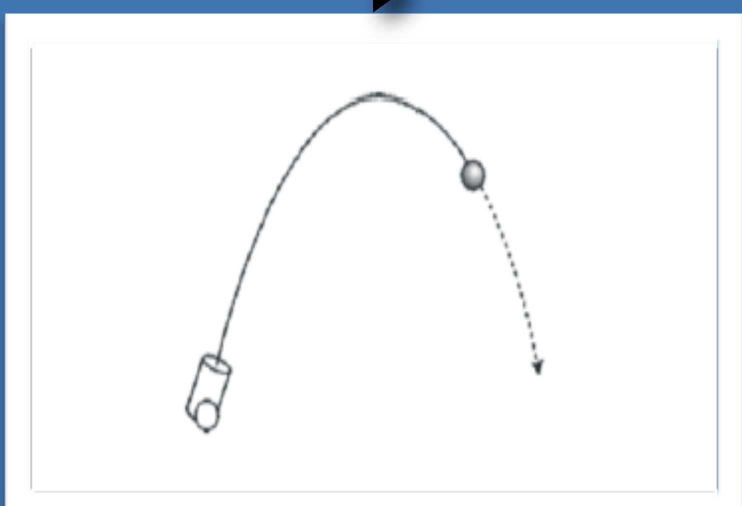
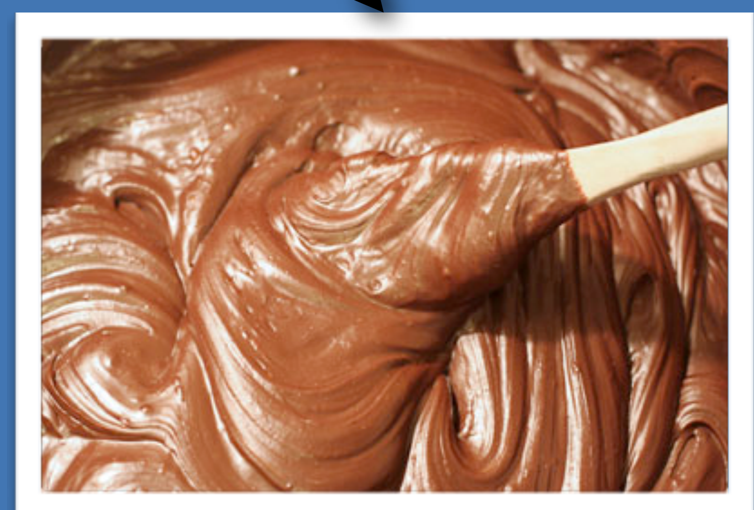


Viscosity

# Fluid Forces

$$\rho\dot{\mathbf{v}} = \rho\mathbf{g} - \nabla p + \mathbf{f}_{\text{viscosity}}$$





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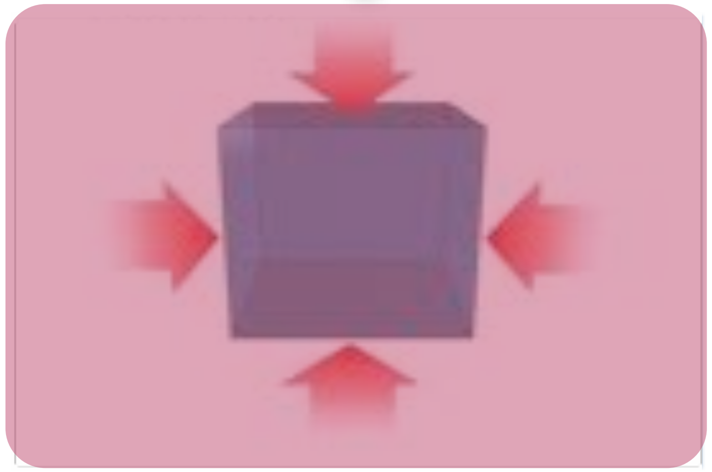

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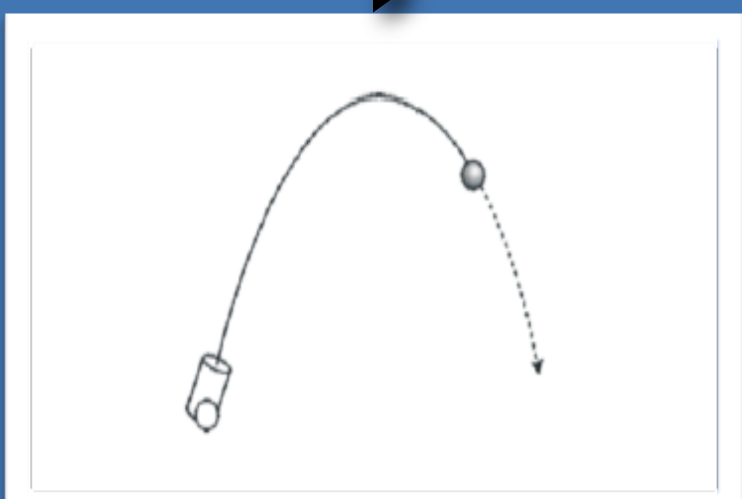
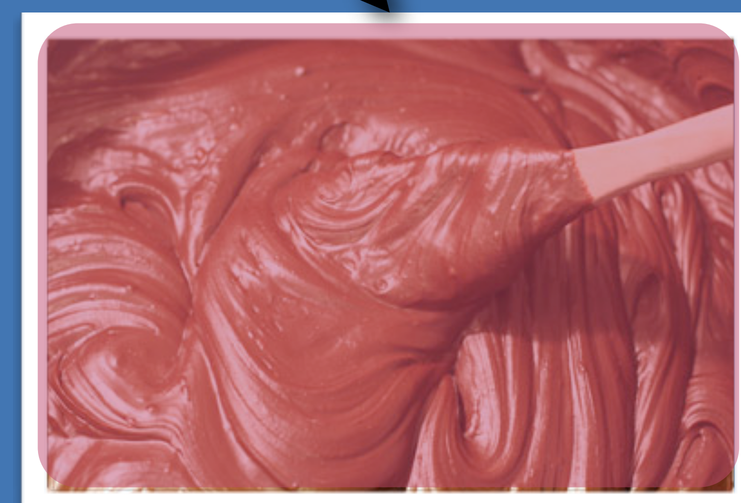


Gravity

Resistance to Compression



Viscosity

# Viscosity

- How do we drive the fluid velocity to be like its neighbors?

- Use a spring!

$$\mathbf{f}_{\text{viscosity}} \propto \underbrace{\mathbf{v}_{\text{neighborhood}}} - \mathbf{v}$$

- What is this?

# Viscosity

$$\mathbf{v}(x-\Delta x) \longleftrightarrow \mathbf{v}(x) \longleftrightarrow \mathbf{v}(x+\Delta x)$$

$$\mathbf{v}^{(1D)}_{\text{neighborhood}} \approx \tfrac{1}{2}\mathbf{v}(x-\Delta x) + \tfrac{1}{2}\mathbf{v}(x+\Delta x)$$

$$\mathbf{v}^{(2D)}_{\text{neighborhood}} \approx \tfrac{1}{4}\mathbf{v}(x-\Delta x) + \tfrac{1}{4}\mathbf{v}(x+\Delta x) + \tfrac{1}{4}\mathbf{v}(x-\Delta y) + \tfrac{1}{4}\mathbf{v}(x+\Delta y)$$

$$\mathbf{f}_{\text{viscosity}} \propto \underbrace{\mathbf{v}_{\text{neighborhood}}}_{} - \mathbf{v}$$

- What is this?

# Viscosity

## By Substitution...

$$f_{\text{viscosity}} \propto \frac{1}{2}\mathbf{v}(x - \Delta x) + \frac{1}{2}\mathbf{v}(x + \Delta x) - \mathbf{v}$$

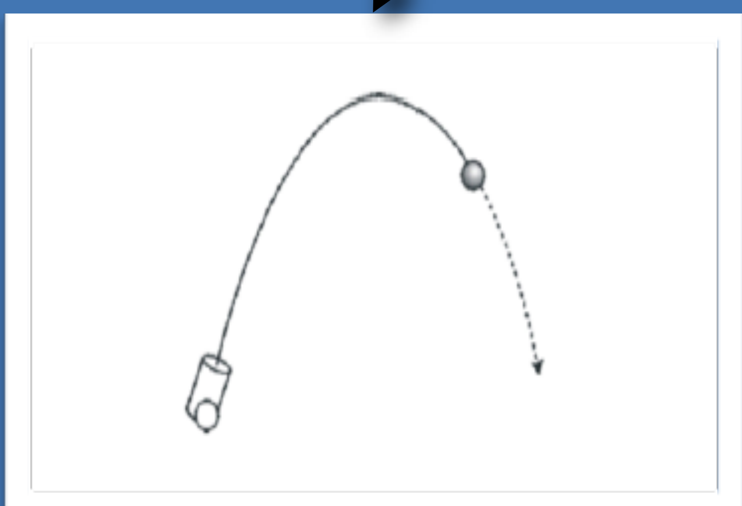
$$f_{\text{viscosity}} \propto \mathbf{v}(x - \Delta x) + \mathbf{v}(x + \Delta x) - 2\mathbf{v}$$

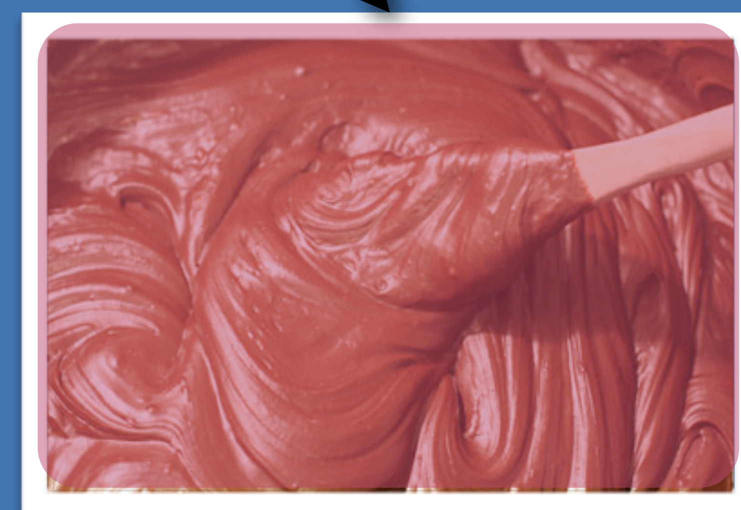
## Taking Limits...

$$f_{\text{viscosity}} = \mu \nabla^2 \mathbf{v}$$

# Fluid Forces

$$\rho\dot{\mathbf{v}} = \rho\mathbf{g} \quad -\nabla p \quad + \mathbf{f}_{\text{viscosity}}$$





Gravity

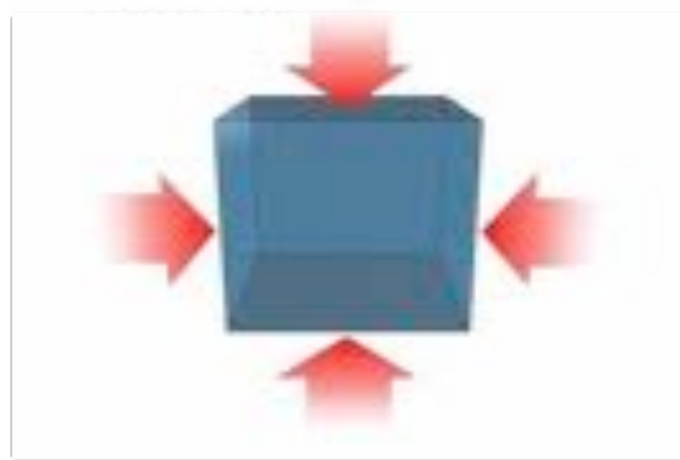
Resistance to Compression

Viscosity

# Fluid Forces

$$\rho\dot{\mathbf{v}} = \rho\mathbf{g} \quad - \nabla p \quad + \mu\nabla^2\mathbf{v}$$

Gravity

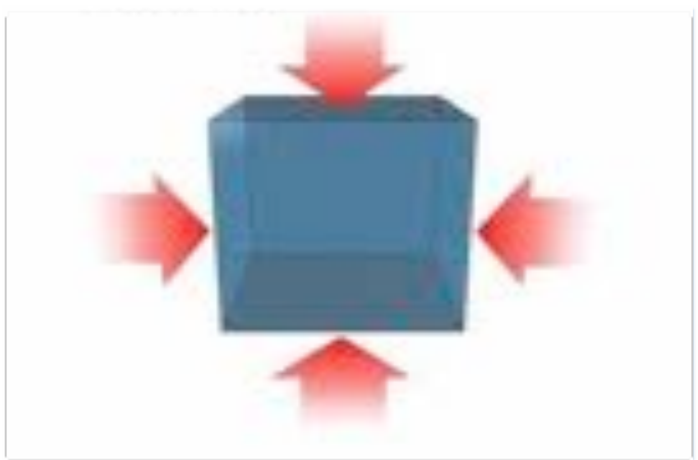


Resistance to Compression

Viscosity

# Navier-Stokes

$$\rho\dot{\mathbf{v}} = \rho\mathbf{g} - \nabla p + \mu\nabla^2\mathbf{v}$$



Gravity

Resistance to Compression

Viscosity

# Navier-Stokes

$$\rho\dot{\mathbf{v}} = \rho\mathbf{g} - \nabla p + \mu\nabla^2\mathbf{v}$$

**How do we evaluate these quantities using a particle system?**

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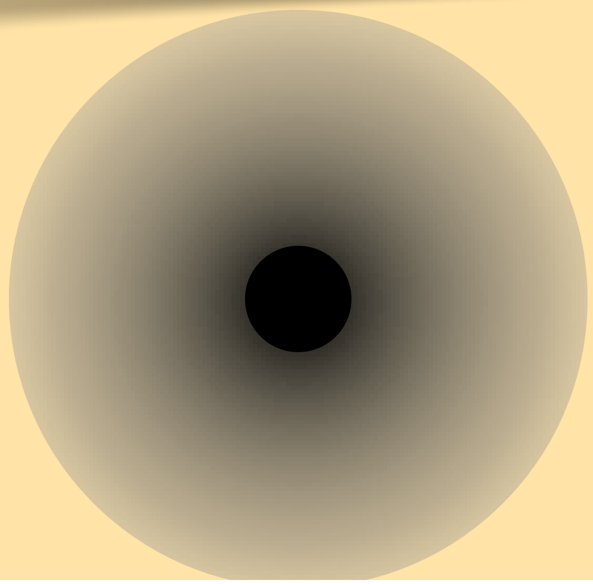
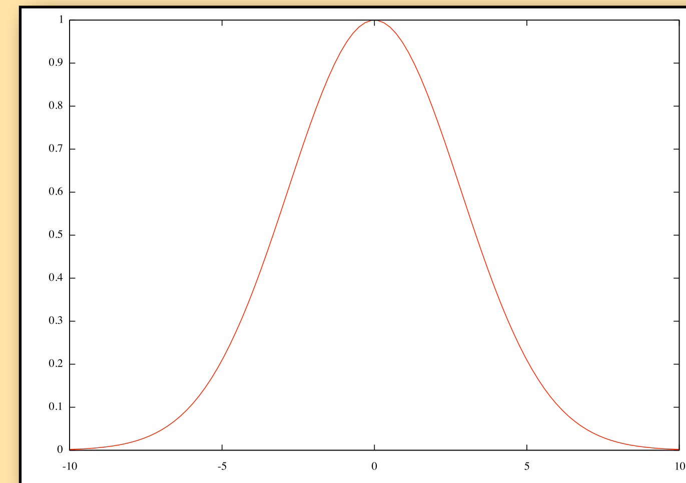
# Kernel Functions



$$W(\mathbf{r})$$

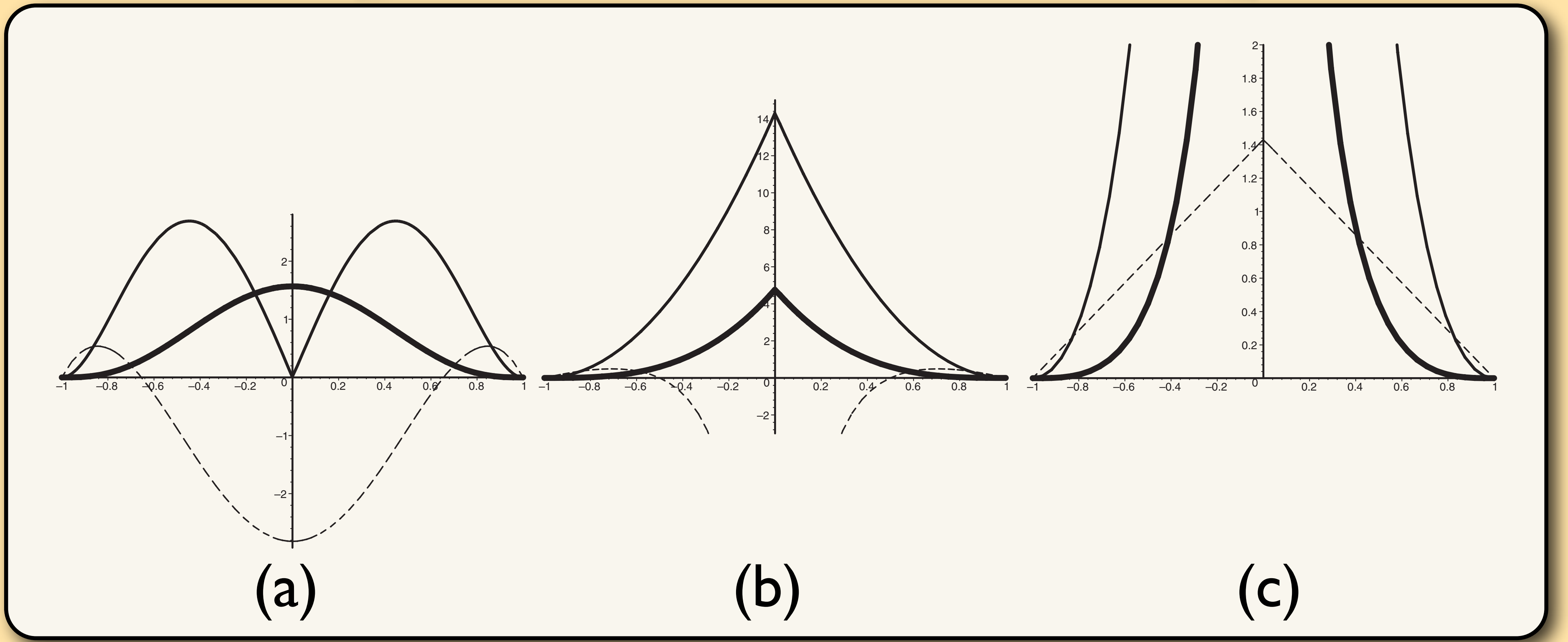
**or**

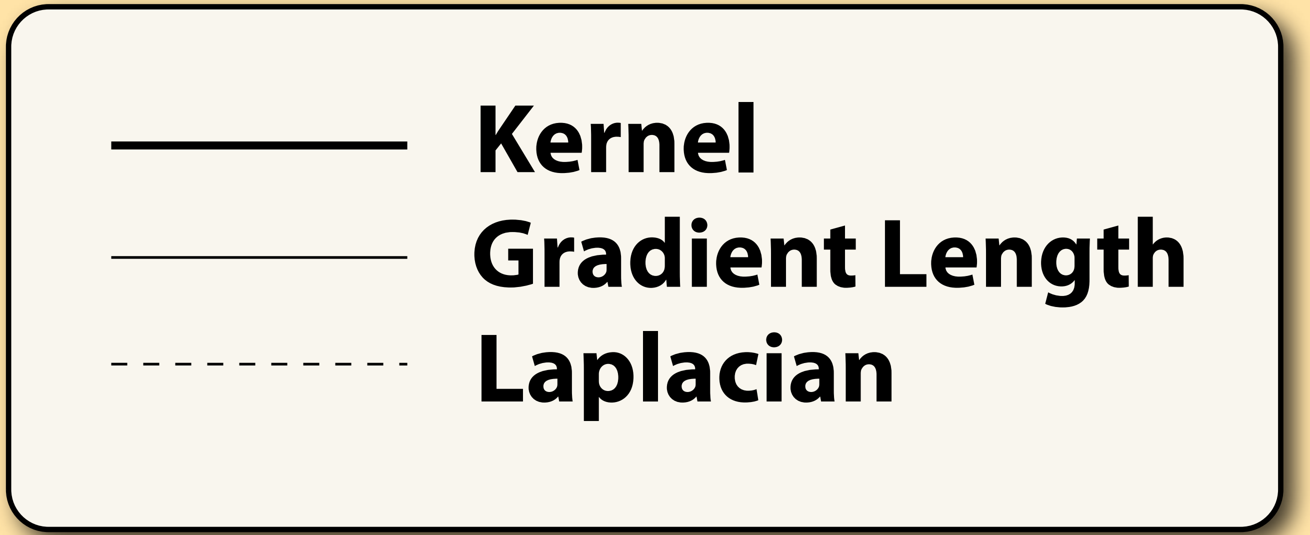
$$W(\mathbf{r}, h)$$



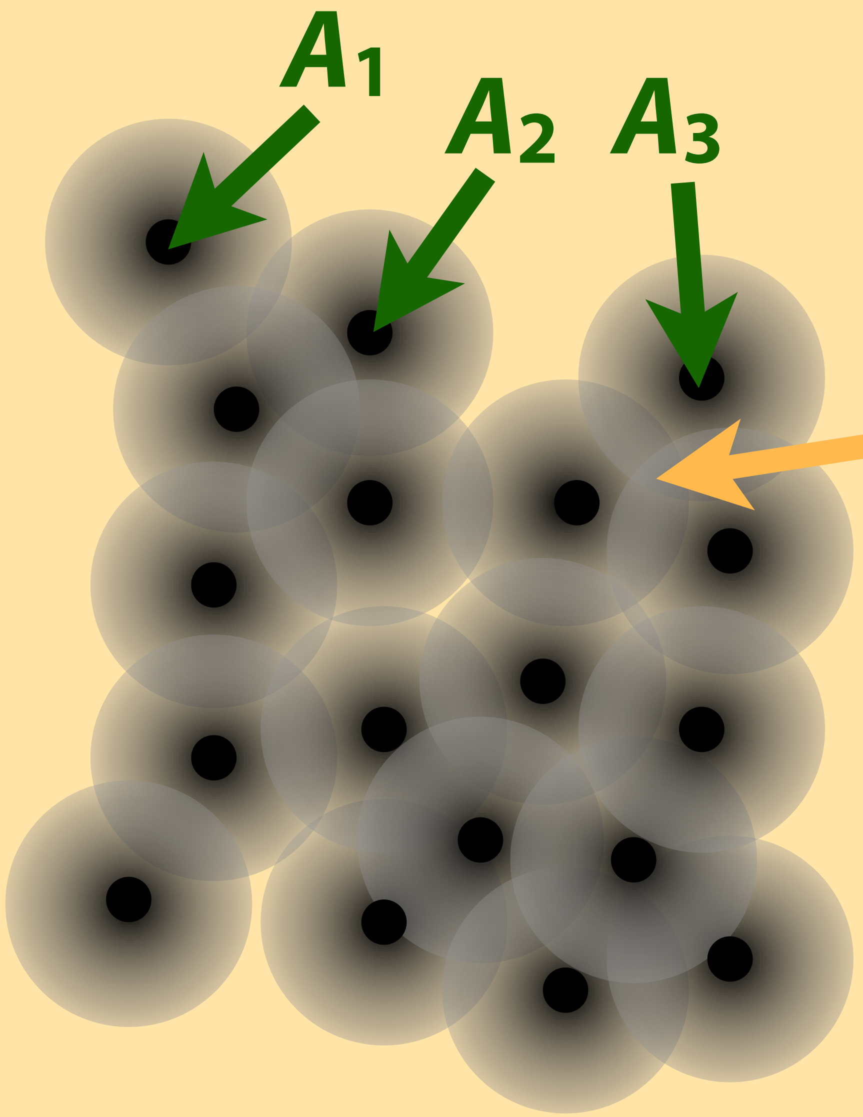

- Properties:
  - Symmetric: $W(\mathbf{x}) = W(-\mathbf{x})$
  - Finite Support: $W(\mathbf{x}) = 0 \ \forall ||x|| > h$
  - Flat at center: $\nabla W(0) = 0$
  - Normalized: $\int W(\mathbf{x}) d\mathbf{x} = 1$

# Kernel



# Interpolation




**For a physical quantity A:**

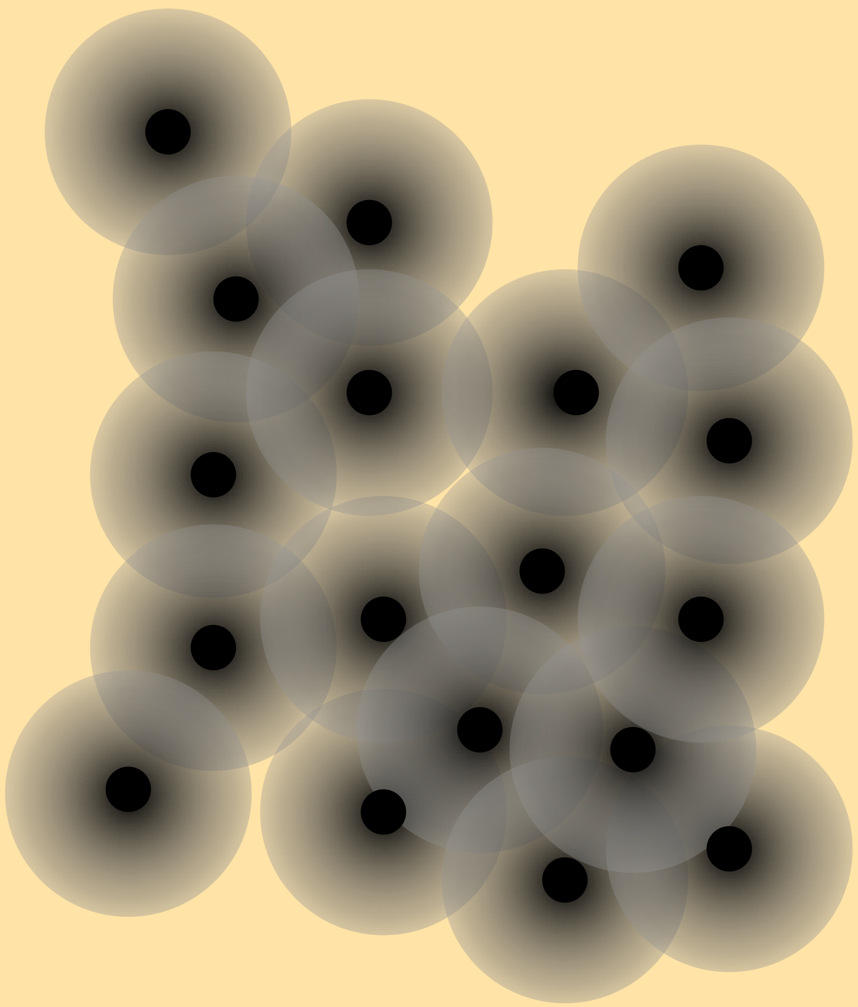
$$A_S(\mathbf{r}) = \sum_j m_j \frac{A_j}{\rho_j} W(\mathbf{r}-\mathbf{r}_j, h),$$



**Example, density:**

$$\rho_S(\mathbf{r}) = \sum_j m_j \frac{\rho_j}{\rho_j} W(\mathbf{r}-\mathbf{r}_j, h)$$

# What About



**Function:**

$$A_S(\mathbf{r}) = \sum_j m_j \frac{A_j}{\rho_j} W(\mathbf{r} - \mathbf{r}_j, h),$$

**Gradient:**

$$\nabla A_S(\mathbf{r}) = \sum_j m_j \frac{A_j}{\rho_j} \nabla W(\mathbf{r} - \mathbf{r}_j, h)$$

**Laplacian:**

$$\nabla^2 A_S(\mathbf{r}) = \sum_j m_j \frac{A_j}{\rho_j} \nabla^2 W(\mathbf{r} - \mathbf{r}_j, h).$$

**But we're going to play tricks. ;-)**

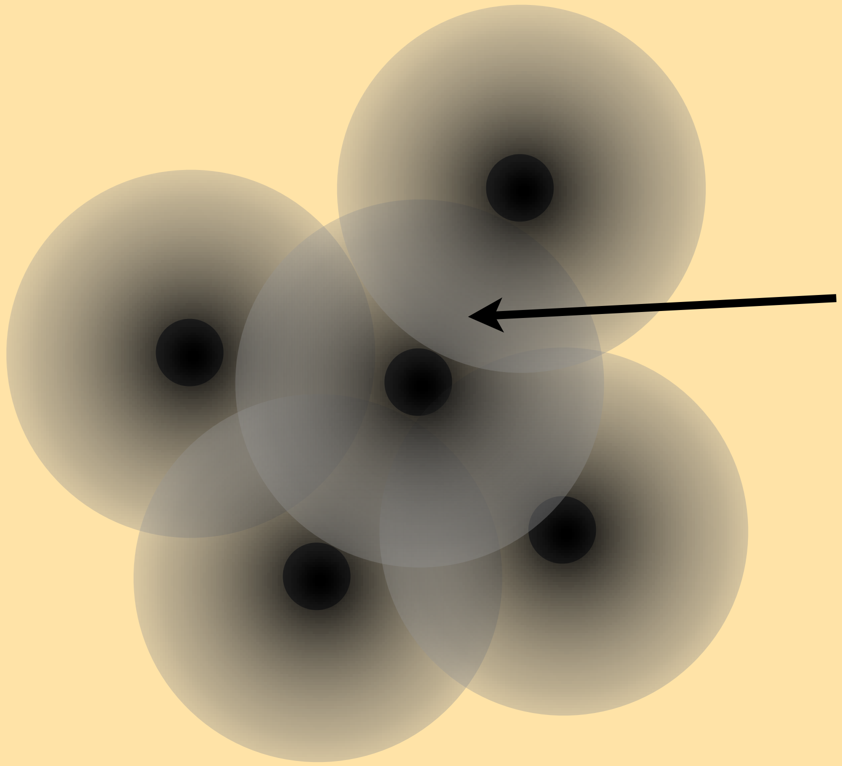
# Pressure

## For One Particle:

$$p_j = \kappa(\rho_j - \overline{\rho})$$

$$\text{pressure force} = -\nabla p(\mathbf{r})$$

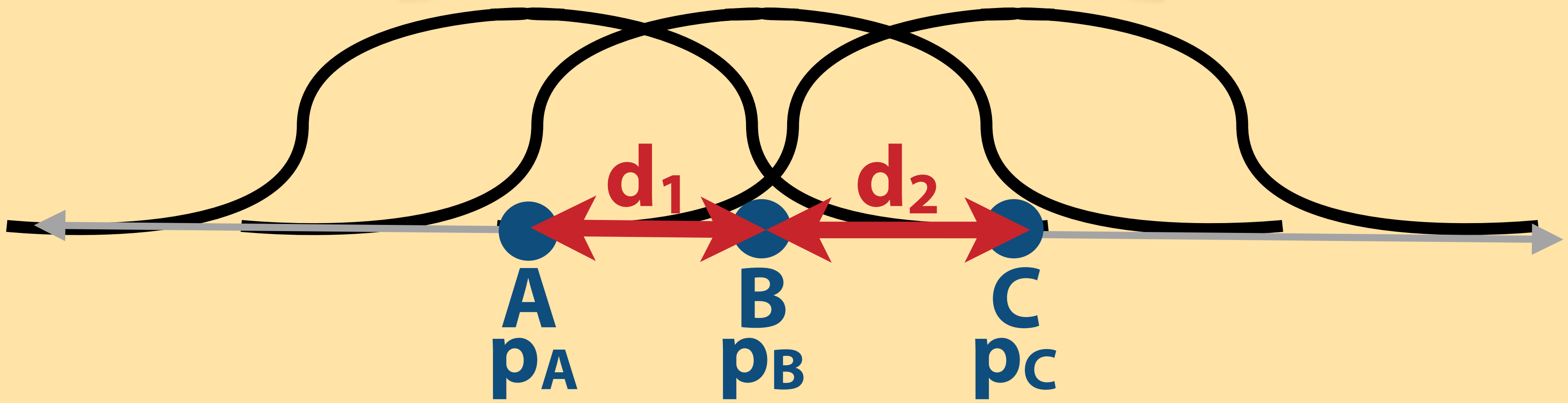
## Spatial Pressure Evaluation:



$$\nabla p(\mathbf{r}) = \sum_j m_j \frac{p_j}{\rho_j} \nabla W(\mathbf{r} - \mathbf{r}_j, h)$$

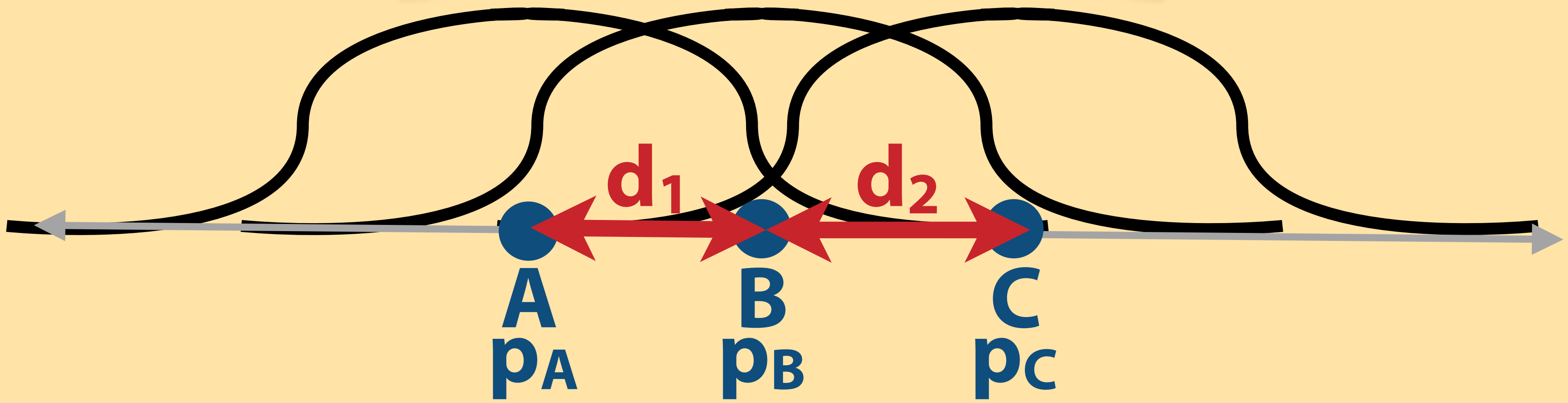
**But wait... is this symmetric?**

# Pressure




$$-\nabla p(\mathbf{r}) = \sum_j p_j \nabla W(\mathbf{r} - \mathbf{r}_j)$$

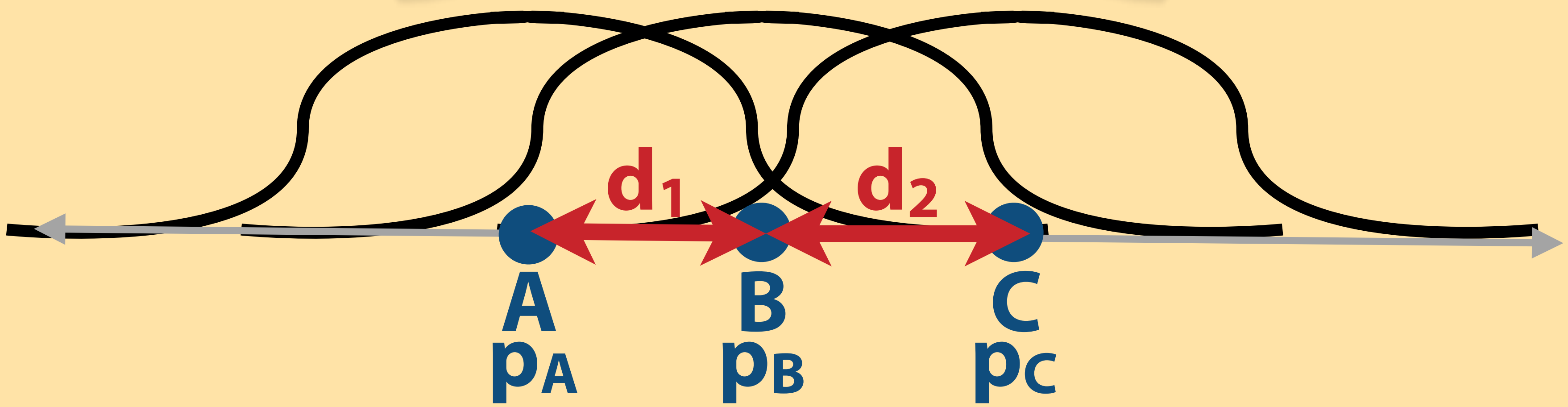
# Pressure




$$-\nabla p(\mathbf{r}) = \sum_j p_j \nabla W(\mathbf{r} - \mathbf{r}_j)$$

$\mathbf{f_A} =$

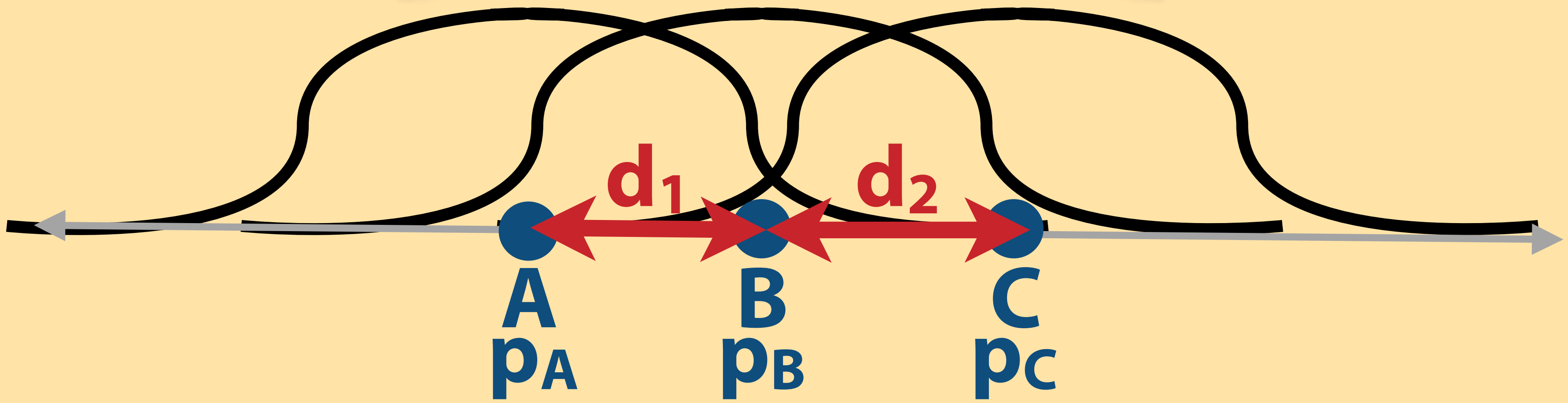
# Pressure




$$-\nabla p(\mathbf{r}) = \sum_j p_j \nabla W(\mathbf{r} - \mathbf{r}_j)$$

$$\mathbf{f_A = -p_A \nabla W(0)}$$

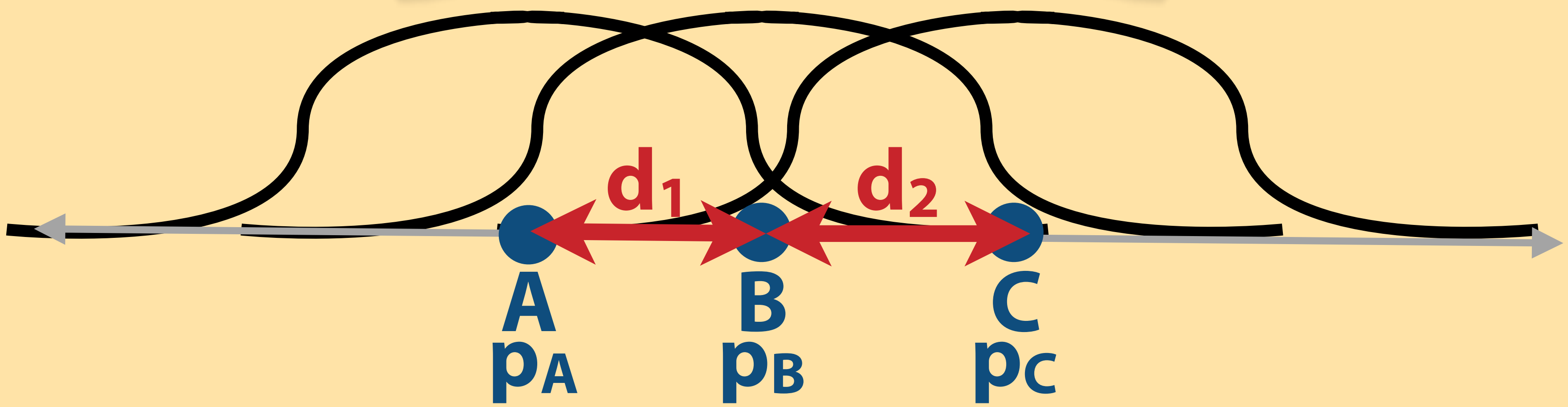
# Pressure




$$-\nabla p(\mathbf{r}) = \sum_j p_j \nabla W(\mathbf{r} - \mathbf{r}_j)$$

$$\mathbf{f_A = -p_A \nabla W(0) - p_B \nabla W(-d_1)}$$

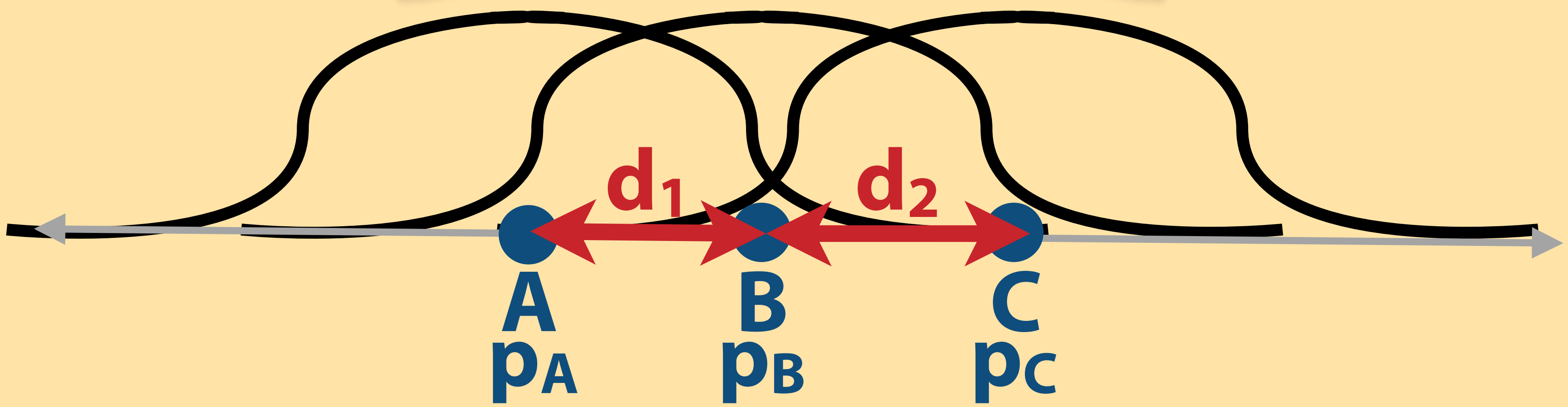
# Pressure




$$-\nabla p(\mathbf{r}) = \sum_j p_j \nabla W(\mathbf{r} - \mathbf{r}_j)$$

$$\mathbf{f_A = -p_A \nabla W(0) - p_B \nabla W(-d_1) - p_C \nabla W(-d_1 - d_2)}$$

# Pressure



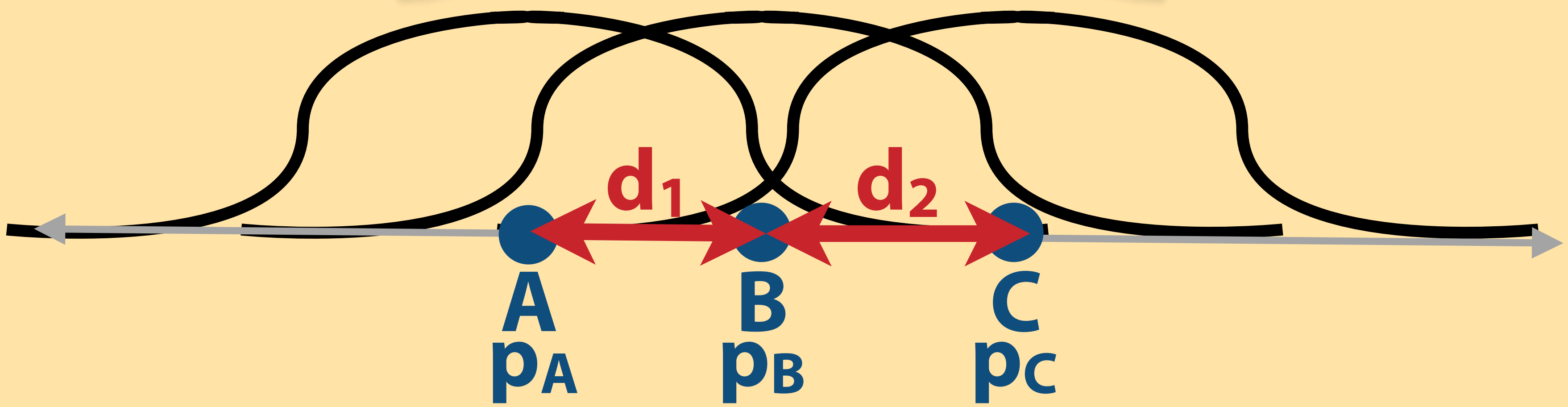

$$-\nabla p(\mathbf{r}) = \sum_j p_j \nabla W(\mathbf{r} - \mathbf{r}_j)$$

$$f_A = -p_A \nabla W(0) - p_B \nabla W(-d_1) - p_C \nabla W(-d_1 - d_2)$$

$$f_B = -p_A \nabla W(d_1) - p_B \nabla W(0) - p_C \nabla W(-d_2)$$

$$f_C = -p_A \nabla W(d_1 + d_2) - p_B \nabla W(d_2) - p_C \nabla W(0)$$

# Pressure

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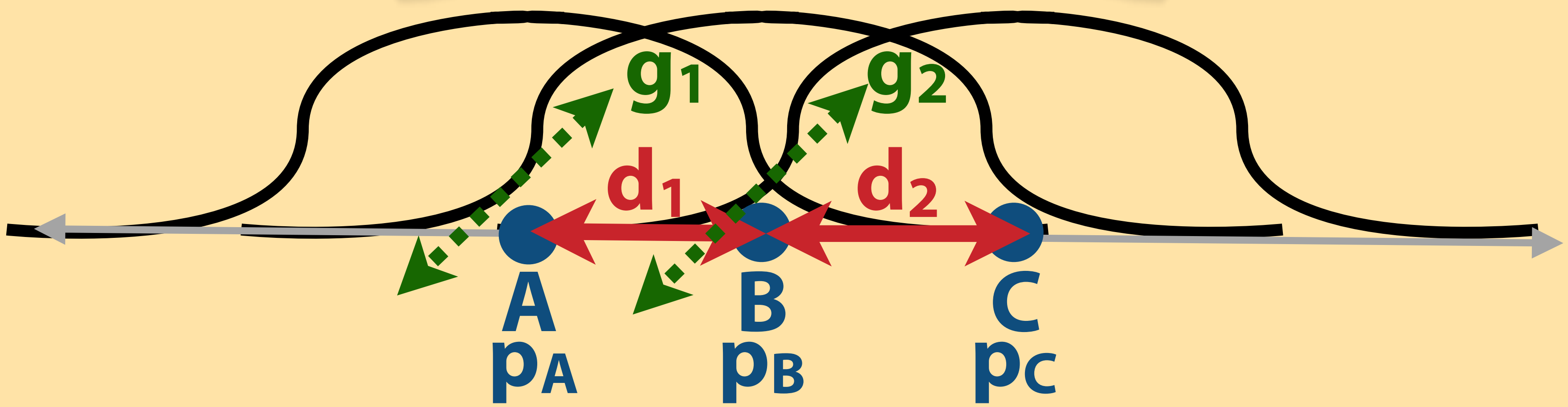
# Pressure




$$-\nabla p(\mathbf{r}) = \sum_j p_j \nabla W(\mathbf{r} - \mathbf{r}_j)$$

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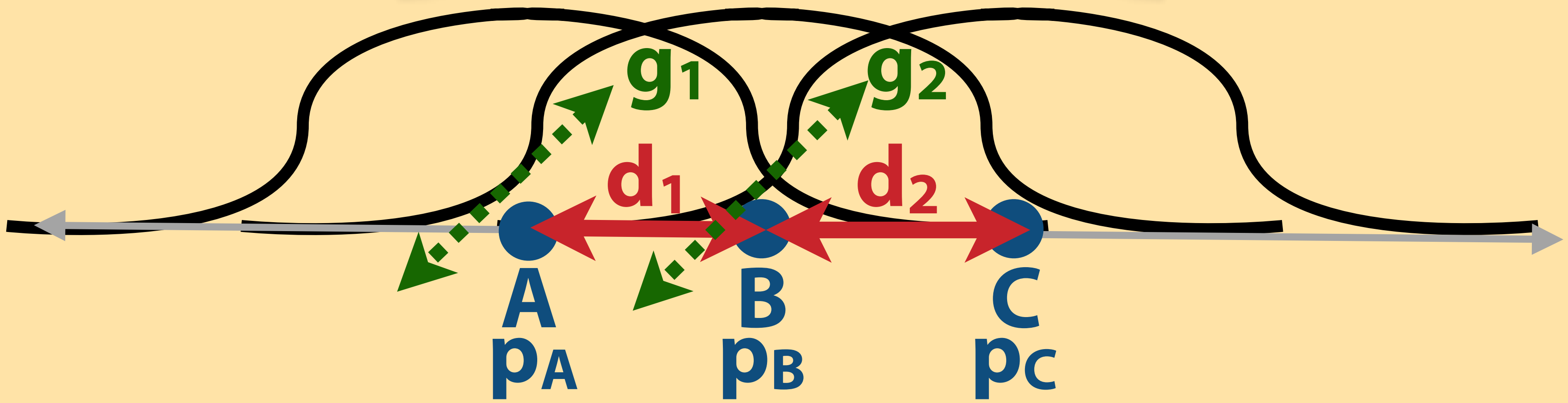
$$f_C = -p_A \nabla W(d_1+d_2) - p_B \nabla W(d_2) - p_C \nabla W(0)$$

$$g_1 = -\nabla W(-d_1) \quad g_2 = -\nabla W(-d_2)$$

# Pressure

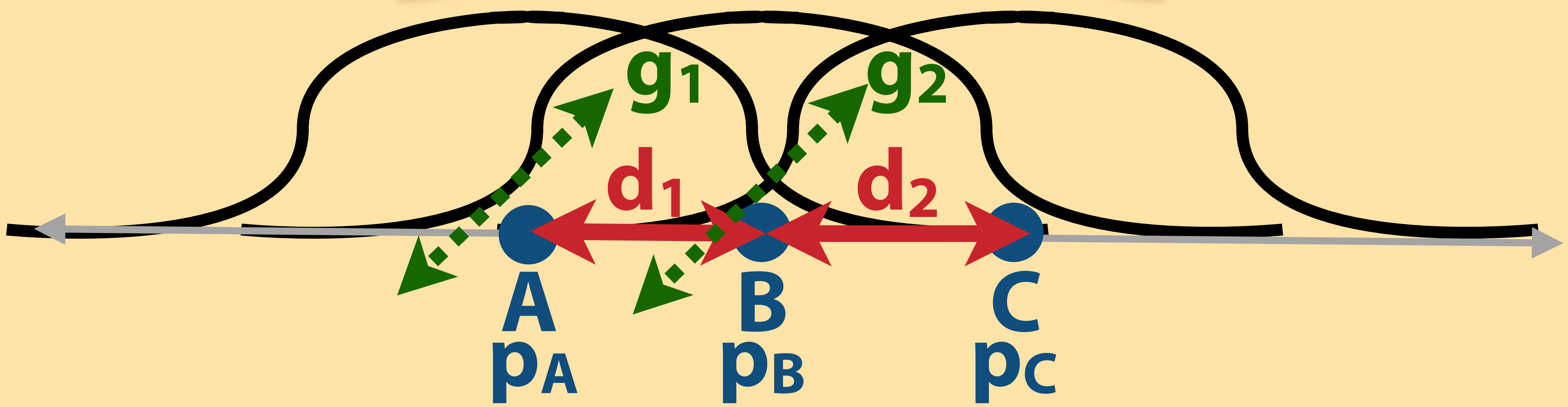



$$-\nabla p(\mathbf{r}) = \sum_j p_j \nabla W(\mathbf{r} - \mathbf{r}_j)$$

$$\left.\begin{aligned} f_A &= p_B\, g_1 \\ f_B &= -p_A\, g_1 + p_C\, g_2 \\ f_C &= -\, p_B\, g_2 \end{aligned}\right\} \neq 0$$

$$g_1 = -\,\nabla W(-d_1) \quad g_2 = -\,\nabla W(-d_2)$$

# Pressure




$$-\nabla p(\mathbf{r}_i) = \sum_j \frac{p_i + p_j}{2} \nabla W(\mathbf{r}_i - \mathbf{r}_j)$$

$$\left.\begin{aligned} f_A &= \tfrac{1}{2}(p_A+p_B)g_1 \\ f_B &= -\tfrac{1}{2}(p_A+p_B)g_1 + \tfrac{1}{2}(p_B+p_C)g_2 \\ f_C &= -\tfrac{1}{2}(p_B+p_C)g_2 \end{aligned}\right\} = 0$$

$$g_1 = -\nabla W(-d_1) \quad g_2 = -\nabla W(-d_2)$$

# Pressure

$$\mathbf{f}_i^{\text{viscosity}} = \mu \nabla^2 \mathbf{v}(\mathbf{r}_a) = \mu \sum_j m_j \frac{\mathbf{v}_j}{\rho_j} \nabla^2 W(\mathbf{r}_i - \mathbf{r}_j, h).$$

## Symmetrization:

$$\mathbf{f}_i^{\text{viscosity}} = \mu \sum_j m_j \frac{\mathbf{v}_j - \mathbf{v}_i}{\rho_j} \nabla^2 W(\mathbf{r}_i - \mathbf{r}_j, h).$$

**Spring that pulls the particle towards the velocity of it's neighbors.**

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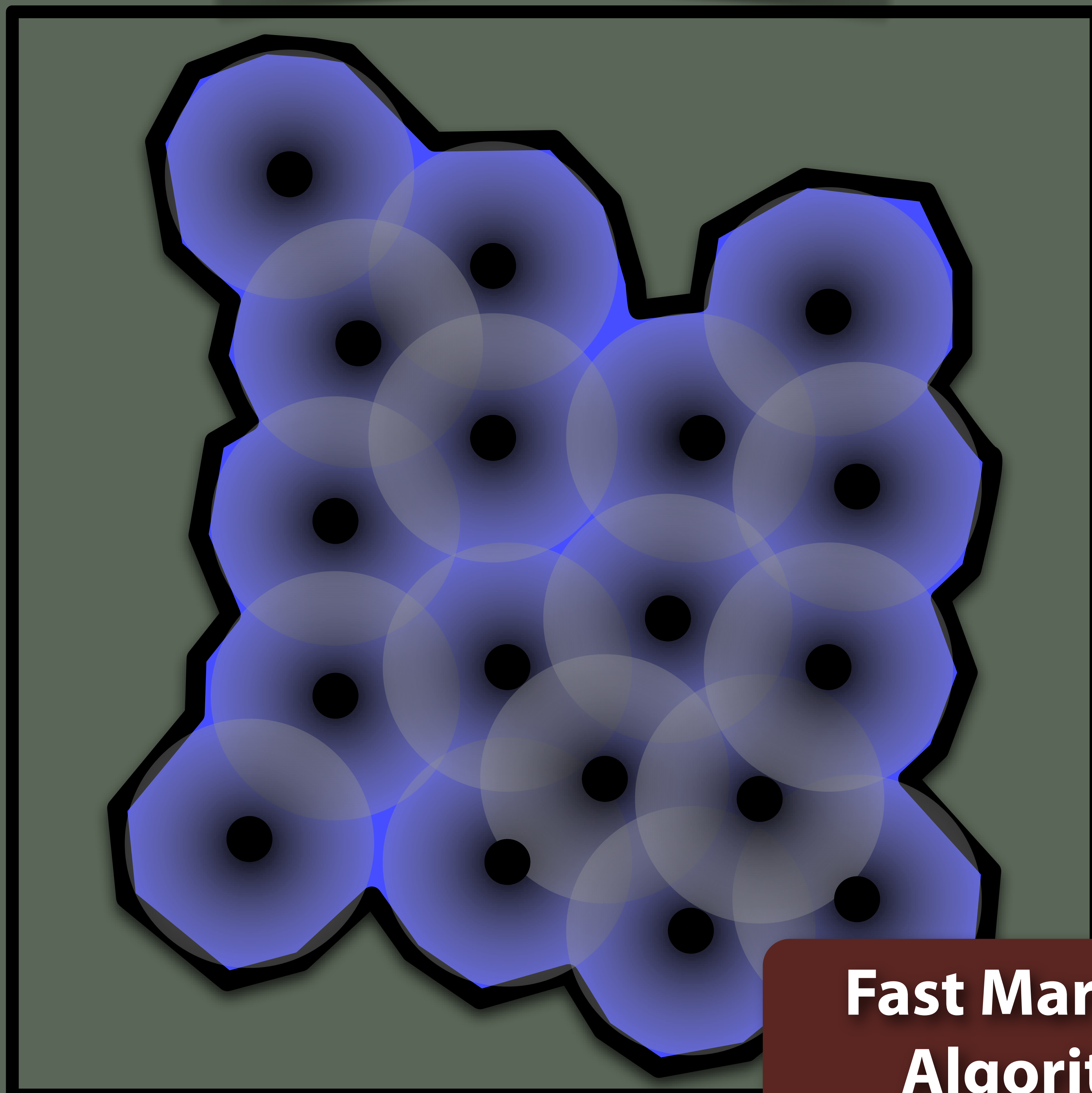
# Rendering



**Fast Marching Algorithm**

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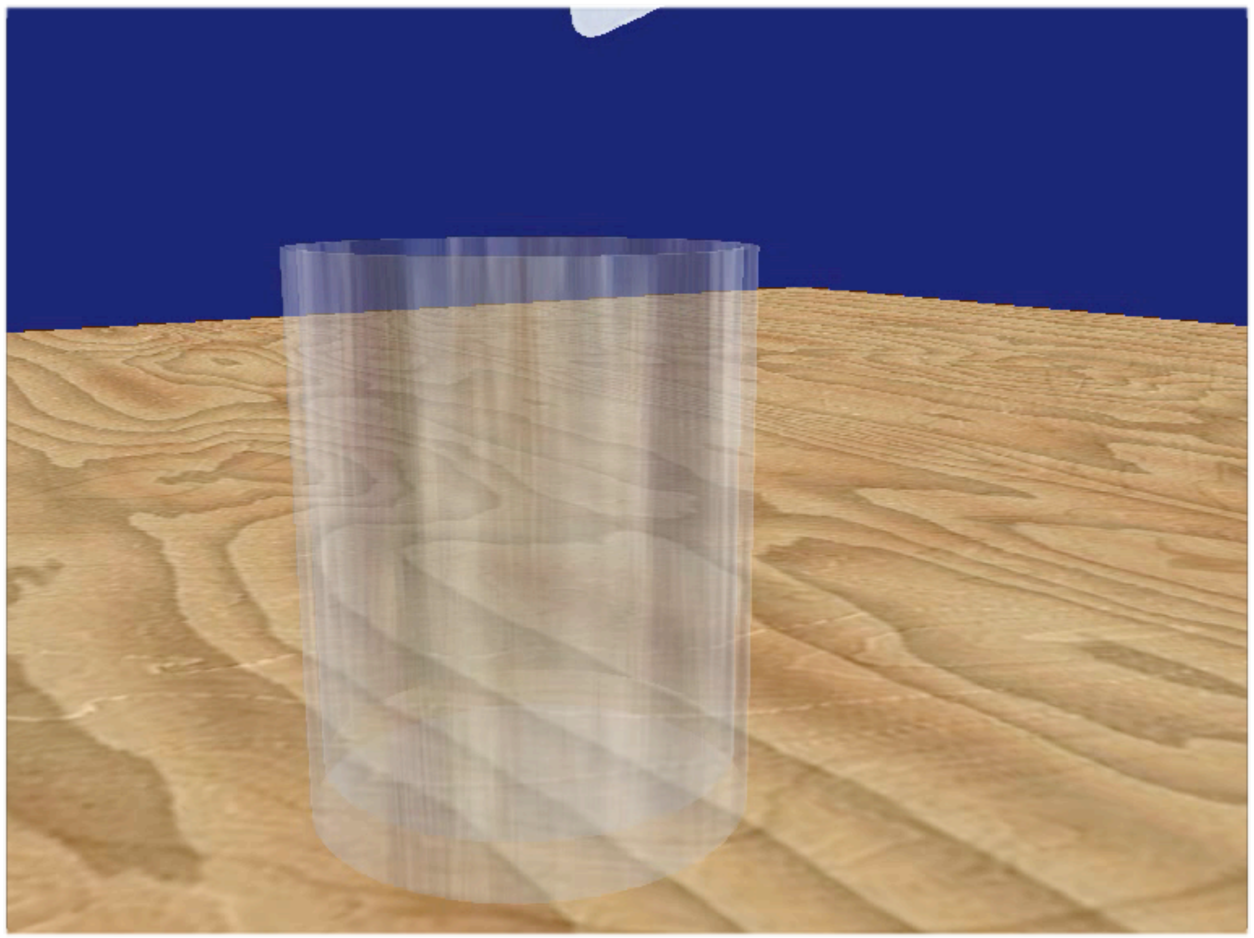
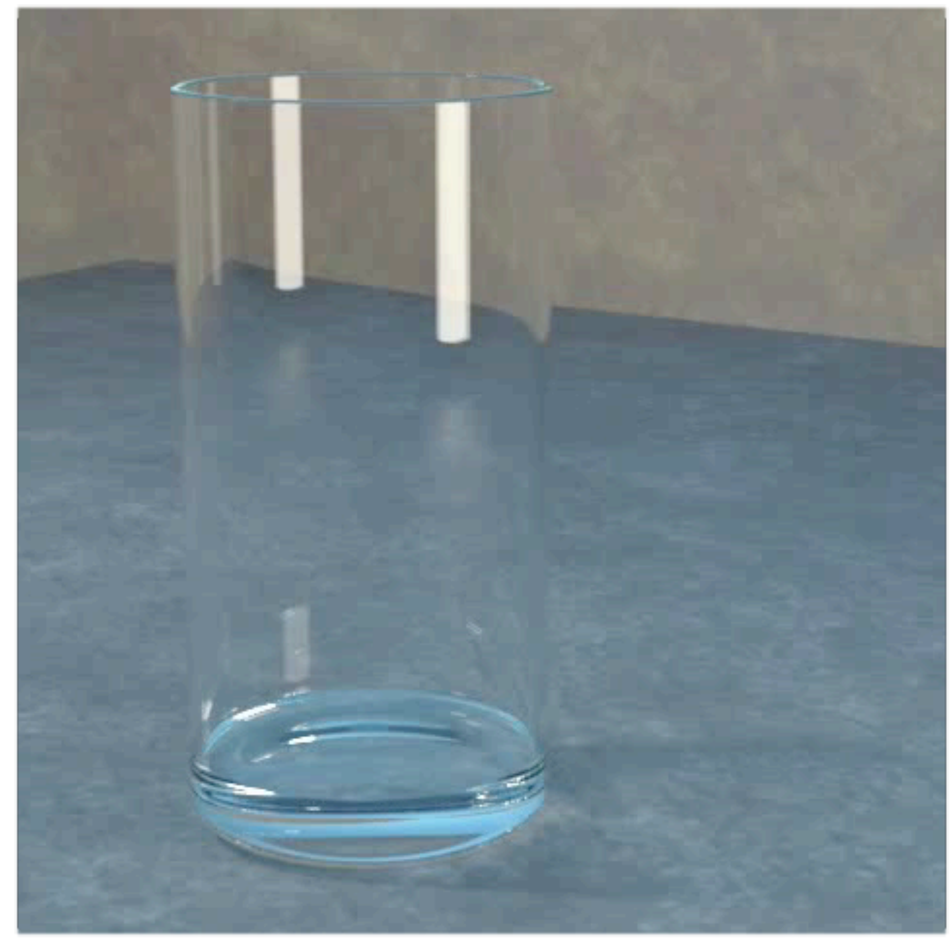
# Example


Predictive-Corrective
Incompressible SPH
Barbara Solenthaler, Renato Pajarola
University of Zurich
VMML
visualization and multimedia lab
university of zürich

# Comparison



SPH



Grid-based

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# Questions

- Which phenomena does the grid-based method capture better? Why?
- Which phenomena does SPH capture better? Why?
- How could we turn our SPH simulator into a grid-based simulator?