Constrained Dynamics: Bead on a Wire

state

$$\frac{d}{dt}\mathbf{x} = \dot{\mathbf{x}}$$

$$\frac{d}{dt}\dot{\mathbf{x}} = \ddot{\mathbf{x}} = \frac{1}{m}\left(\mathbf{f} + \hat{\mathbf{f}}\right)$$

constraints

$$C = \frac{1}{2}\left(\mathbf{x}\cdot\mathbf{x} - 1\right) = 0$$

$$\dot{C} = \mathbf{x}\cdot\dot{\mathbf{x}} = 0$$

At any point the set of *legal velocities* are those which are perpendicular to **x**. Conversely, the *illegal velocities* are parallel to **x** i.e. λ**x**.

$$\ddot{C} = \dot{\mathbf{x}}\cdot\dot{\mathbf{x}} + \mathbf{x}\cdot\ddot{\mathbf{x}} = 0$$

$$= \dot{\mathbf{x}}\cdot\dot{\mathbf{x}} + \mathbf{x}\cdot\left(\frac{1}{m}\left(\mathbf{f} + \hat{\mathbf{f}}\right)\right)$$

$$\therefore \quad \frac{1}{m}\hat{\mathbf{f}}\cdot\mathbf{x} = -\dot{\mathbf{x}}\cdot\dot{\mathbf{x}} - \frac{1}{m}\mathbf{x}\cdot\mathbf{f}$$

virtual work

$$T = \frac{1}{2}m\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}$$

$$\dot{T} = m\dot{\mathbf{x}}\cdot\ddot{\mathbf{x}}$$

$$\dot{T} = \dot{\mathbf{x}}\cdot\left(\mathbf{f} + \hat{\mathbf{f}}\right)$$

$$\dot{T}\big|_{\text{due to }\hat{\mathbf{f}}} = \dot{\mathbf{x}}\cdot\hat{\mathbf{f}} = 0$$

Since the constraint force is perpendicular to all legal velocities, it must be of the form λ**x**.

$$\therefore \quad \hat{\mathbf{f}} = \lambda\mathbf{x}$$

therefore

$$\frac{\lambda}{m}\mathbf{x}\cdot\mathbf{x} = -\dot{\mathbf{x}}\cdot\dot{\mathbf{x}} - \frac{1}{m}\mathbf{x}\cdot\mathbf{f}$$

$$\lambda = m\left(\frac{-\dot{\mathbf{x}}\cdot\dot{\mathbf{x}} - \frac{1}{m}\mathbf{x}\cdot\mathbf{f}}{\mathbf{x}\cdot\mathbf{x}}\right)$$

Constrained Dynamics: General Case

state

$$\frac{d}{dt}\mathbf{x} = \dot{\mathbf{x}}$$

$$\frac{d}{dt}\dot{\mathbf{x}} = \ddot{\mathbf{x}} = M^{-1}\left(\mathbf{f} + \hat{\mathbf{f}}\right) = W\left(\mathbf{f} + \hat{\mathbf{f}}\right)$$

constraints

$$\mathbf{C}(\mathbf{x}) = \mathbf{0}$$

$$\dot{\mathbf{C}} = \frac{d\mathbf{C}}{dt} = \frac{\partial \mathbf{C}}{\partial \mathbf{x}}\dot{\mathbf{x}} = J\dot{\mathbf{x}} = \mathbf{0}$$

$$\ddot{\mathbf{C}} = \dot{J}\dot{\mathbf{x}} + J\ddot{\mathbf{x}} = \mathbf{0}$$

$$= \dot{J}\dot{\mathbf{x}} + JW\left(\mathbf{f} + \hat{\mathbf{f}}\right)$$

$$\therefore \; JW\hat{\mathbf{f}} = -\dot{J}\dot{\mathbf{x}} - JW\mathbf{f}$$

At any point the set of *legal velocities* are those which are perpendicular to the rows of J. Conversely, the *illegal velocities* are spanned by J^T i.e. $\{J^T\lambda \mid \lambda \in \mathbf{R}^c\}$.

virtual work

$$T = \frac{1}{2}\dot{\mathbf{x}}^T M \dot{\mathbf{x}}$$

$$\dot{T} = \dot{\mathbf{x}}^T M \ddot{\mathbf{x}}$$

$$\dot{T} = \dot{\mathbf{x}} \cdot \left(\mathbf{f} + \hat{\mathbf{f}}\right)$$

$$\dot{T}\big|_{\text{due to }\hat{\mathbf{f}}} = \dot{\mathbf{x}} \cdot \hat{\mathbf{f}} = 0$$

$$\therefore \; \hat{\mathbf{f}} = J^T\lambda$$

Since the constraint force is perpendicular to all legal velocities, it must be in the span of J^T.

therefore

$$JWJ^T\lambda = -\dot{J}\dot{\mathbf{x}} - JW\mathbf{f}$$

$$\therefore \; \lambda = \left(JWJ^T\right)^{-1}\left(-\dot{J}\dot{\mathbf{x}} - JW\mathbf{f}\right)$$