Inverse Problems

$$\vec{f}(\vec{x}) = \vec{y} \quad \leftarrow \text{solve}$$

$$\vec{f}(\vec{x}) \approx \vec{f}(\vec{x}_0) + \vec{\nabla}\vec{f}(\vec{x}_0)(\vec{x} - \vec{x}_0) \approx \vec{y}$$

$$\tilde{y} \approx \left(\nabla f|_{x_0}\right)^{-1}\left(\vec{y} - \vec{f}(\vec{x}_0)\right) + \vec{x}_0$$

Optimization Problems

$$\vec{f}(\vec{x}) \quad \leftarrow \text{find the minimum.}$$

$$\nabla \vec{f}(\vec{x}) = 0$$

$$M\vec{x} = \vec{b} \quad (\text{if } f \text{ is quadratic})$$

Example

$$\phi(x_1, x_2 \cdots x_5) = \frac{1}{2}(x_1 - 1)^2 + \frac{1}{2}(x_1 - x_2)^2 + \frac{1}{2}(x_2 - x_3)^2 + \frac{1}{2}(x_3 - x_4)^2 + \frac{1}{2}(x_4 - x_5)^2 + \frac{1}{2}(x_5 - 2)^2$$

$$\frac{d\phi}{dx_1} = (x_1 - 1) + (x_1 - x_2) = 0 \qquad 2x_1 - x_2 = 1$$

$$\frac{d\phi}{dx_2} = -(x_1 - x_2) + (x_2 - x_3) = 0 \qquad -2x_2 - x_1 - x_3 = 0$$

$$\frac{d\phi}{dx_3} = -(x_2 - x_3) + (x_3 - x_4) = 0 \qquad 2x_3 - x_2 - x_4 = 0$$

$$\frac{d\phi}{dx_4} = -(x_3 - x_4) + (x_4 - x_5) = 0 \qquad 2x_4 - x_3 - x_5 = 0$$

$$\frac{d\phi}{dx_5} = -(x_4 - x_5) + (x_5 - 2) = 0 \qquad 2x_5 - x_4 = 2$$

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$