DiffEQ 1
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DiffEQ Integration

Differential Equation Basics

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Pixar Animation Studios
A Canonical Differential Equation

\[ \dot{x} = f(x, t) \]

- \( x(t) \): a moving point.
- \( f(x,t) \): \( x \)'s velocity.
The differential equation
\[
\dot{x} = f(x, t)
\]
defines a vector field over \( x \).
Integral Curves

Pick any starting point, and follow the vectors.
Given the starting point, follow the integral curve.
Euler’s Method

- Simplest numerical solution method
- Discrete time steps
- Bigger steps, bigger errors.

\[ x(t + \Delta t) = x(t) + \Delta t f(x, t) \]
Two Problems

- Accuracy
- Instability
Accuracy

Consider the equation:

\[ \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x \]

What do the integral curves look like?
Error turns $x(t)$ from a circle into the spiral of your choice.
Problem 2: Instability

- Consider the following system:

\[
\begin{align*}
\dot{x} &= -x \\
x(0) &= 1
\end{align*}
\]
Problem 2: Instability

Euler's Method

\[ x(t + \Delta t) = x(t) + \Delta t f(x, t) \]

Simplest numerical solution method

Discrete time steps

Bigger steps, bigger errors.

Problem I: Inaccuracy

Error turns \( x(t) \) from a circle into the spiral of your choice.

Problem II: Instability to Neptune!
Accuracy of Euler Method

\[ \dot{x} = f(x) \]

Consider Taylor Expansion about \( x(t) \)...

\[
x(t + h) = x(t) + hf(x(t)) + O(h^2)
\]

Therefore, Euler’s method has error \( O(h^2) \)... it is first order.

How can we get to \( O(h^3) \) error?
The Midpoint Method

- Also known as second order Runge-Kutta:

\[ k_1 = h(f(x_0, t_0)) \]

\[ k_2 = hf(x_0 + \frac{k_1}{2}, t_0 + \frac{h}{2}) \]

\[ x(t_0 + h) = x_0 + k_2 + O(h^3) \]
The Midpoint Method

a. Compute an Euler step
\[ \Delta x = \Delta t f(x, t) \]
b. Evaluate \( f \) at the midpoint
\[ f_{\text{mid}} = f\left(\frac{x + \Delta x}{2}, \frac{t + \Delta t}{2}\right) \]
c. Take a step using the midpoint value
\[ x(t + \Delta t) = x(t) + \Delta t f_{\text{mid}} \]
q-Stage Runge-Kutta

General Form:

\[ x(t_0 + h) = x_0 + h \sum_{i=1}^{q} w_i k_i \]

where:

\[ k_i = f \left( x_0 + h \sum_{j=1}^{i-1} \beta_{ij} k_j \right) \]

Find the constant that ensure accuracy \( O(h^n) \).
4th-Order Runge-Kutta

\[ k_1 = hf(x_0, t_0) \]

\[ k_2 = hf(x_0 + \frac{k_1}{2}, t_0 + \frac{h}{2}) \]

\[ k_3 = hf(x_0 + \frac{k_2}{2}, t_0 + \frac{h}{2}) \]

\[ k_4 = hf(x_0 + k_3, t_0 + h) \]

\[ x(t_0 + h) = x_0 + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4 + O(h^5) \]

Why so popular?
<table>
<thead>
<tr>
<th>Order</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stages</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>7</td>
<td>9</td>
<td>11</td>
</tr>
</tbody>
</table>
More methods...

- Euler’s method is 1st Order.
- The midpoint method is 2nd Order.
- Just the tip of the iceberg. See *Numerical Recipes* for more.
- Helpful hints:
  - *Don’t* use Euler’s method (you will anyway.)
  - *Do* use adaptive step size.
Modular Implementation

- Generic operations:
  - Get dim(x)
  - Get/set x and t
  - Deriv Eval at current (x,t)
- Write solvers in terms of these.
  - Re-usable solver code.
  - Simplifies model implementation.
Solver Interface

- Dim(state)
- Get/Set State
- Deriv Eval

System

Solver
void eulerStep(Sys sys, float h) {
    float t = getTime(sys);
    vector<float> x0, deltaX;

    t = getTime(sys);
    x0 = getState(sys);
    deltaX = derivEval(sys, x0, t);
    setState(sys, x0 + h*deltaX, t+h);
}

Question

• How could we model the solar system with differential equations?

• What would the variables be?

• What would the equations be (roughly)?

• What sorts of common physical phenomena are not well modeled by differential equations?
Student Answers

How can we model the solar system?

- each variable is a planet location + velocity:
  - 6 * n variables for n planets
- second order differential equation (define acceleration)
- need to know mass, volume, and several other variables (SOME OF CONSTANT - OR DERIVABLE)
- simplifying assumption: only depends on the sun!
- to compute all force pairs - need n^2 computation

What sorts of phenomena are not well modeled?

- things that travel very fast
  - light (travels "instantaneously")
  - electricity
- phenomena which happen much faster than "timestep"
  - collisions (DISCONTINUITIES!)
- free will
  - human motion (!!!)
- tearing, molecular properties of a material