DiffEQ 1

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DiffEQ Integration

Differential Equation Basics

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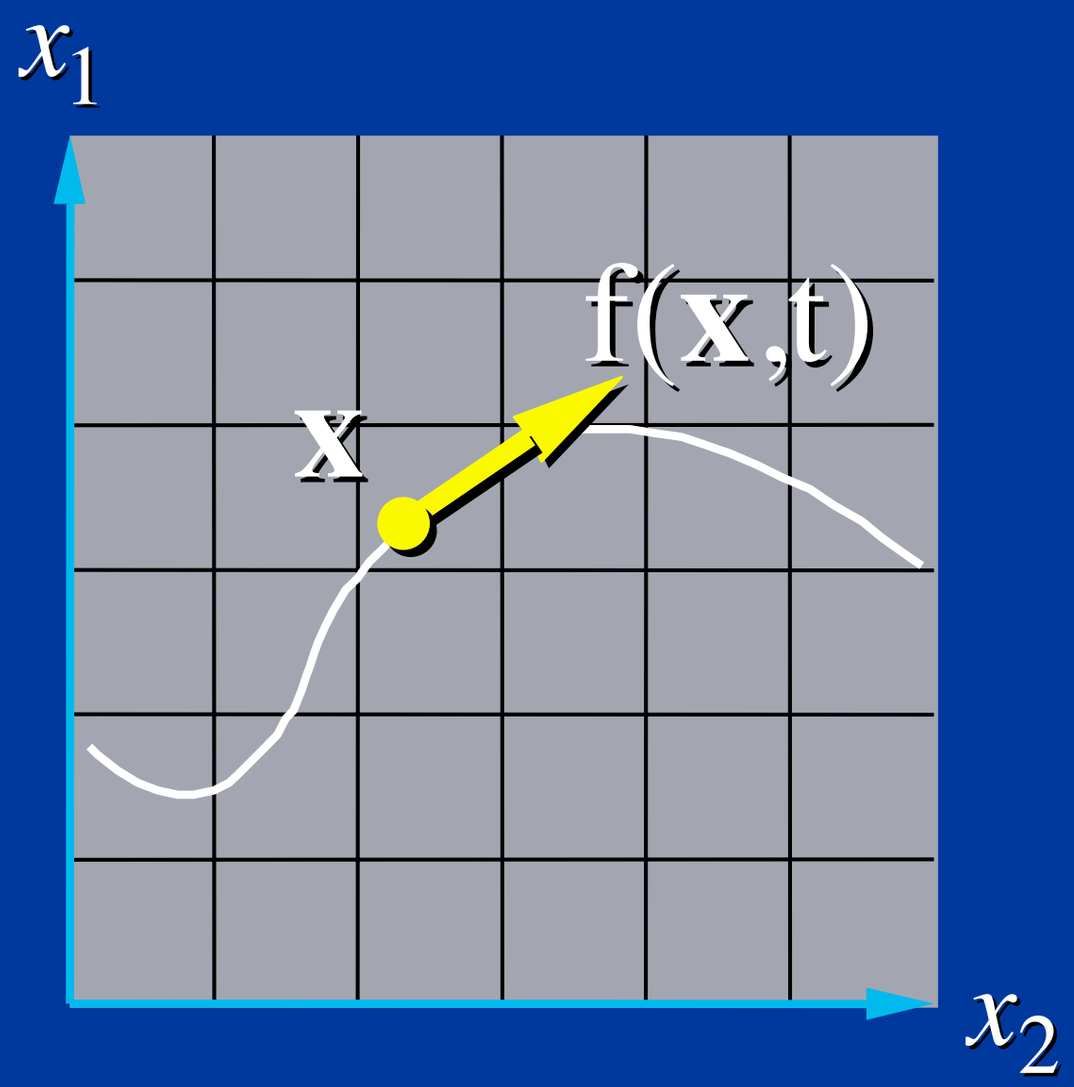
PIXAR
ANIMATION STUDIOS

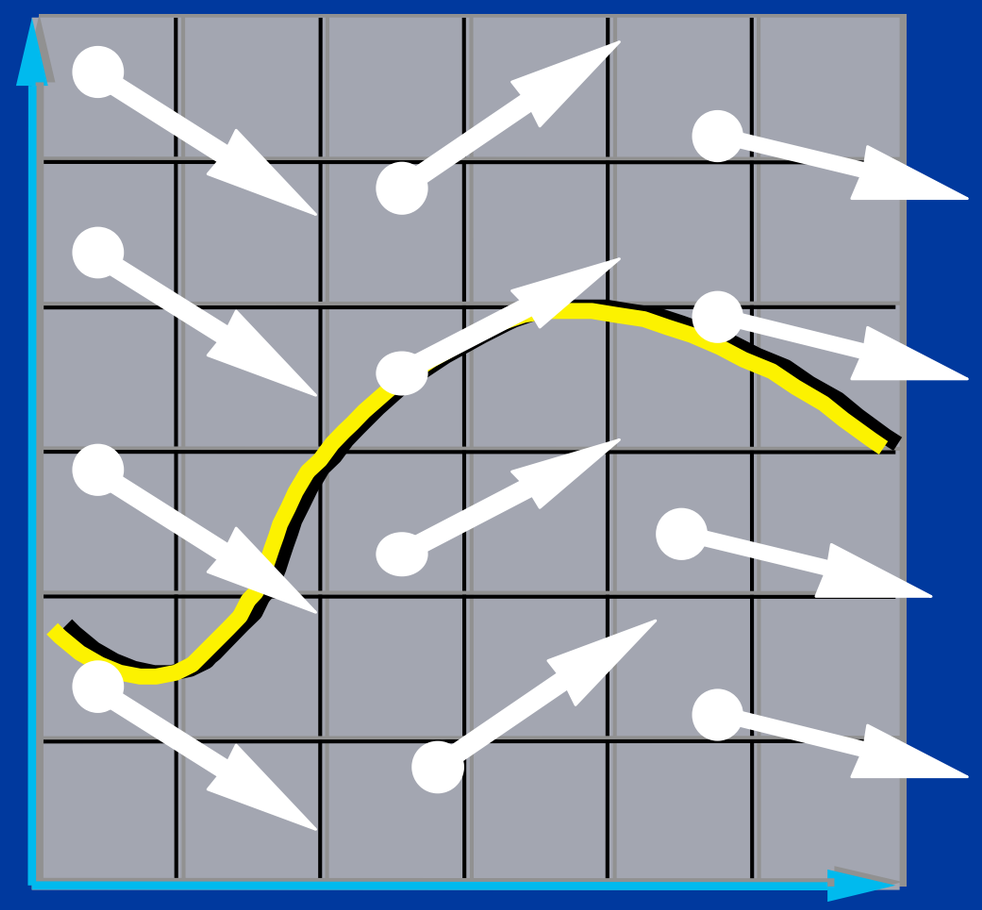
A Canonical Differential Equation



$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

- **x(t): a moving point.**
- **f(x,t): x's velocity.**

Vector Field



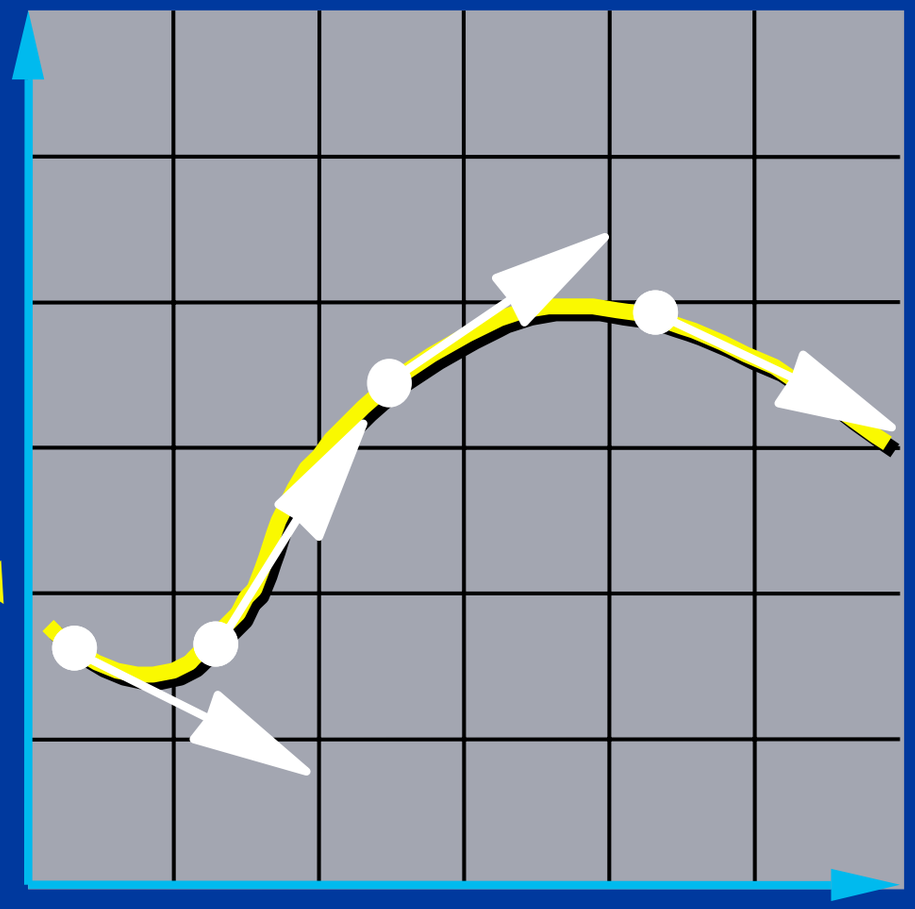
The differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

defines a vector field over x.

Integral Curves

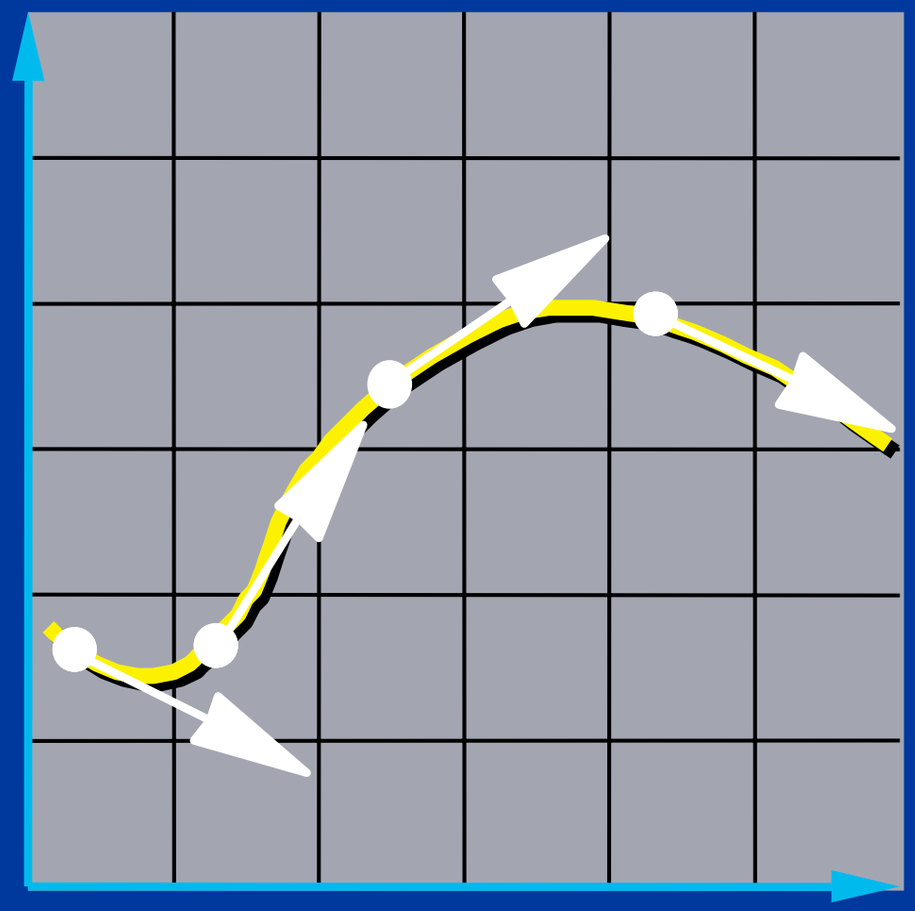
Start Here



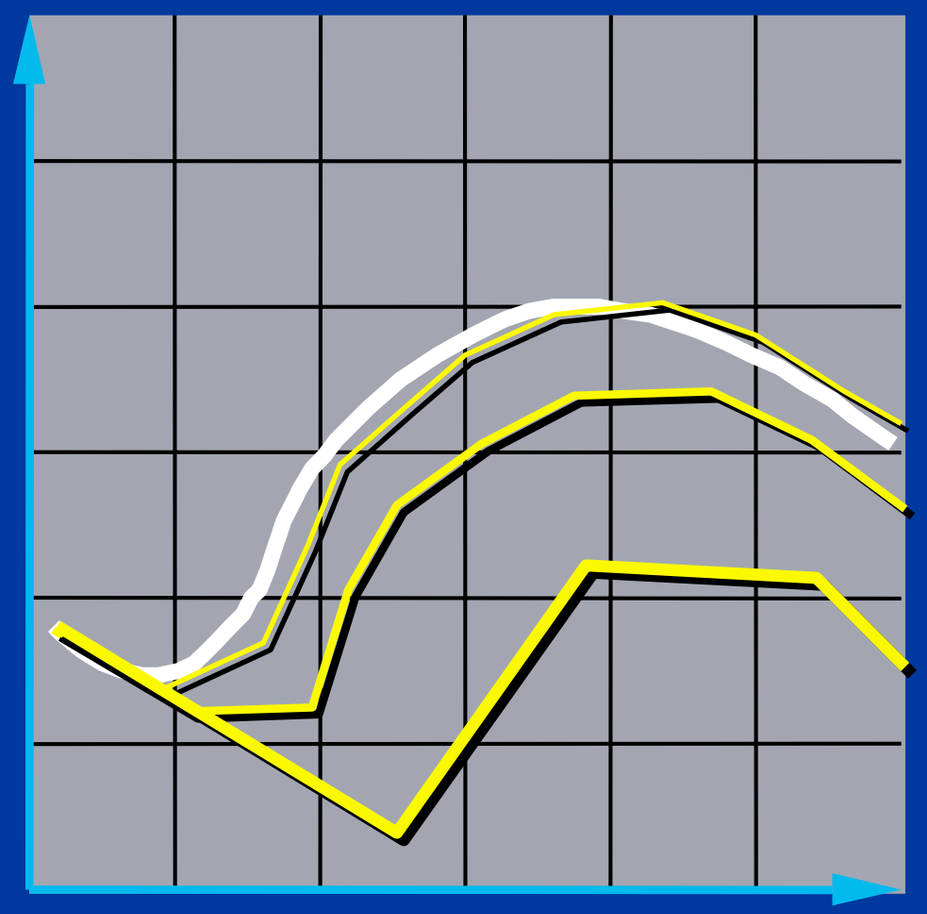
Pick any starting point,
and follow the vectors.

Initial Value Problems

Given the starting point, follow the integral curve.



Euler's Method



- Simplest numerical solution method
- Discrete time steps
- Bigger steps, bigger errors.

$$\mathbf{x}(t + \Delta t) = \mathbf{x}(t) + \Delta t\, \mathbf{f}(\mathbf{x}, t)$$

Two Problems

- Accuracy
- Instability

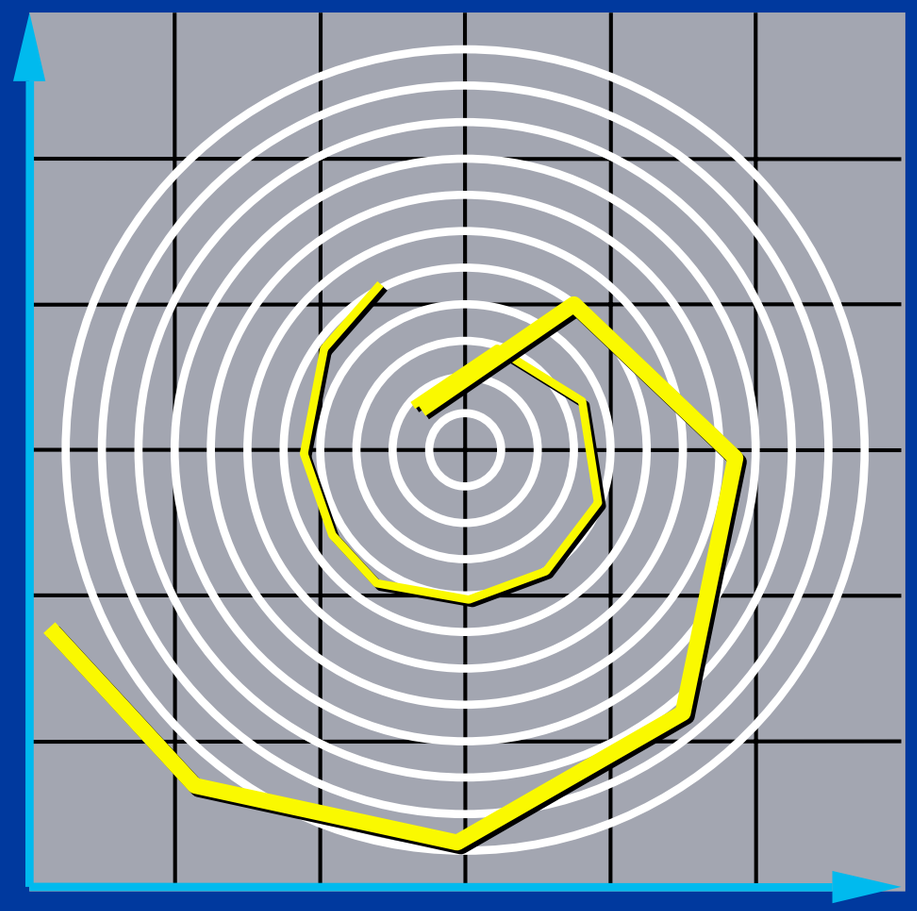
Accuracy

Consider the equation:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}$$

What do the integral curves look like?

Problem I: Inaccuracy



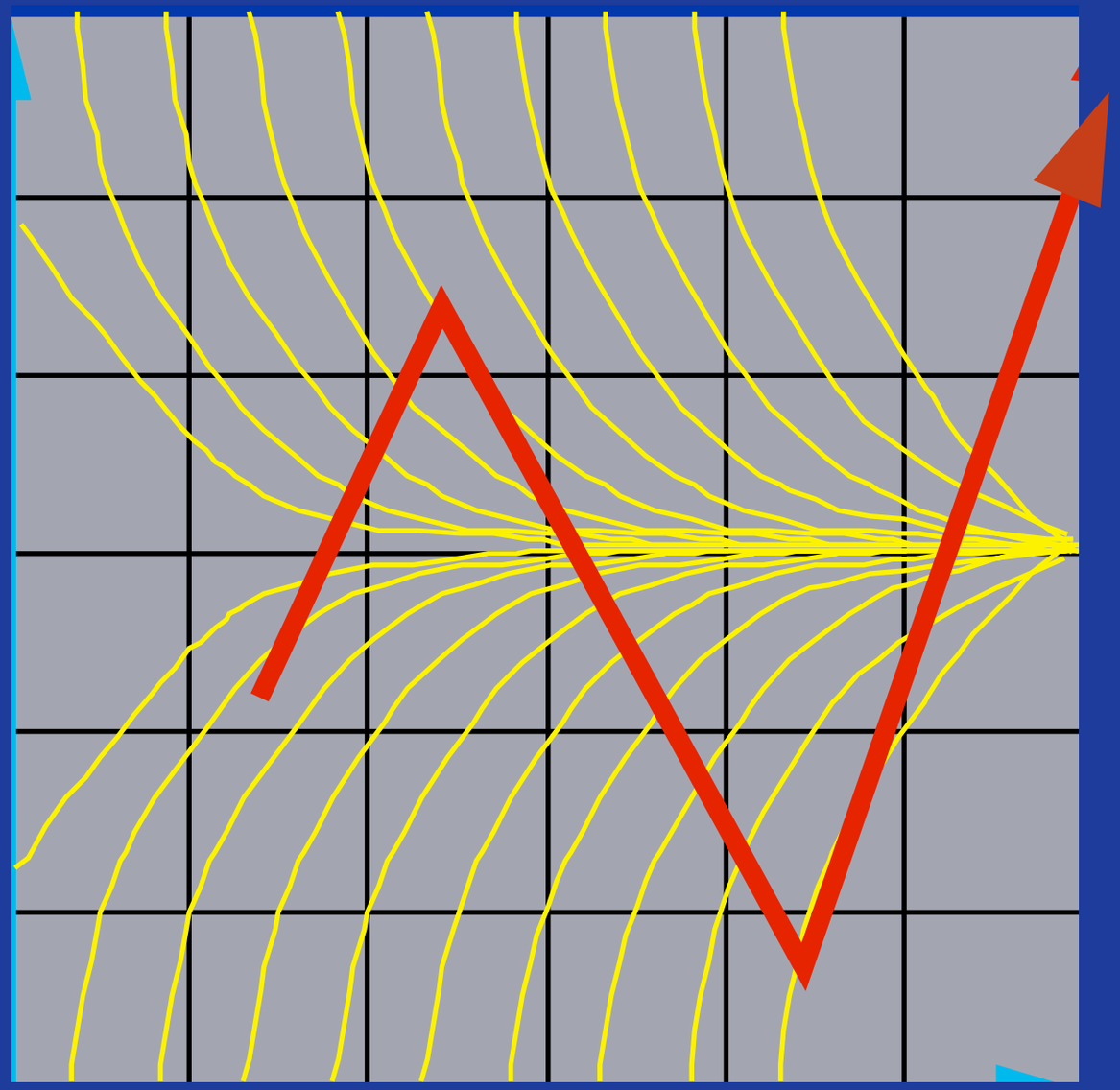
Error turns x(t) from a circle into the spiral of your choice.

Problem 2: Instability

- Consider the following system:

$$\begin{cases} \dot{x} = -x \\ x(0) = 1 \end{cases}$$

Problem 2: Instsability



To Neptune!

Accuracy of Euler Method

$$\dot{x} = f(x)$$

Consider Taylor Expansion about x(t)...

$$x(t+h) = \overbrace{\underbrace{x(t)}_{\text{constant}} + \underbrace{hf(x(t))}_{\text{linear}}}^{\text{Euler step}} + \overbrace{\underbrace{O(h^2)}_{\text{everything else}}}^{\text{error}}$$

Therefore, Euler's method has error O (h^2)... it is *first order.*

How can we get to O(h^3) error?

The Midpoint Method

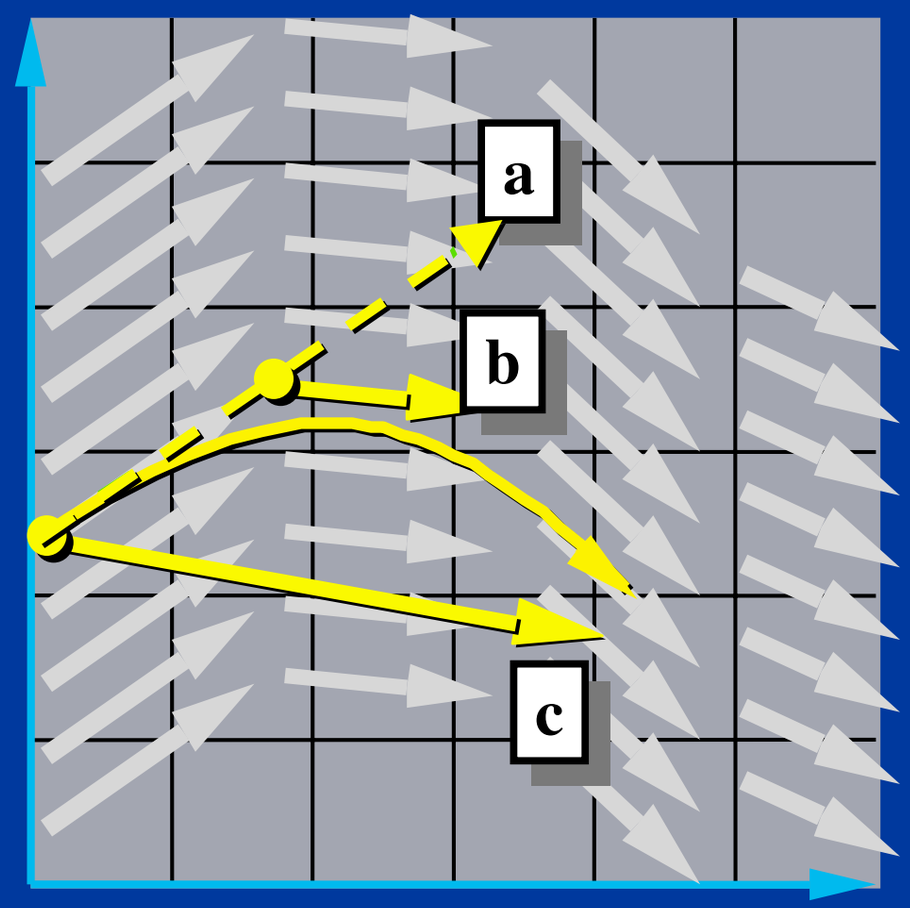
- **Also known as second order Runge-Kutte:**

$$k_1 = h(f(x_0, t_0)$$

$$k_2 = hf(x_0 + \frac{k_1}{2}, t_0 + \frac{h}{2})$$

$$x(t_0 + h) = x_0 + k_2 + O(h^3)$$

The Midpoint Method



a. Compute an Euler step

$$\Delta \mathbf{x} = \Delta t\, \mathbf{f}(\mathbf{x}, t)$$

b. Evaluate f at the midpoint

$$\mathbf{f}_{\text{mid}} = \mathbf{f}\left(\frac{\mathbf{x} + \Delta \mathbf{x}}{2}, \frac{t + \Delta t}{2}\right)$$

c. Take a step using the midpoint value

$$\mathbf{x}(t + \Delta t) = \mathbf{x}(t) + \Delta t\, \mathbf{f}_{\text{mid}}$$

q-Stage Runge-Kutta

General Form:

$$x(t_0 + h) = x_0 + h \sum_{i=1}^{q} w_i k_i$$

where:

$$k_i = f\left(x_0 + h \sum_{j=1}^{i-1} \beta_{ij} k_j \right)$$

Find the constant that ensure accuracty $O(h^n)$.

4th-Order Runge-Kutta

$$k_1 = hf(x_0, t_0)$$

$$k_2 = hf(x_0 + \frac{k_1}{2}, t_0 + \frac{h}{2})$$

$$k_3 = hf(x_0 + \frac{k_2}{2}, t_0 + \frac{h}{2})$$

$$k_4 = hf(x_0 + k_3, t_0 + h)$$

$$x(t_0 + h) = x_0 + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4 + O(h^5)$$

Why so popular?

Order vs. Stages

Order	1	2	3	4	5	6	7	8
Stages	1	2	3	4	6	7	9	11

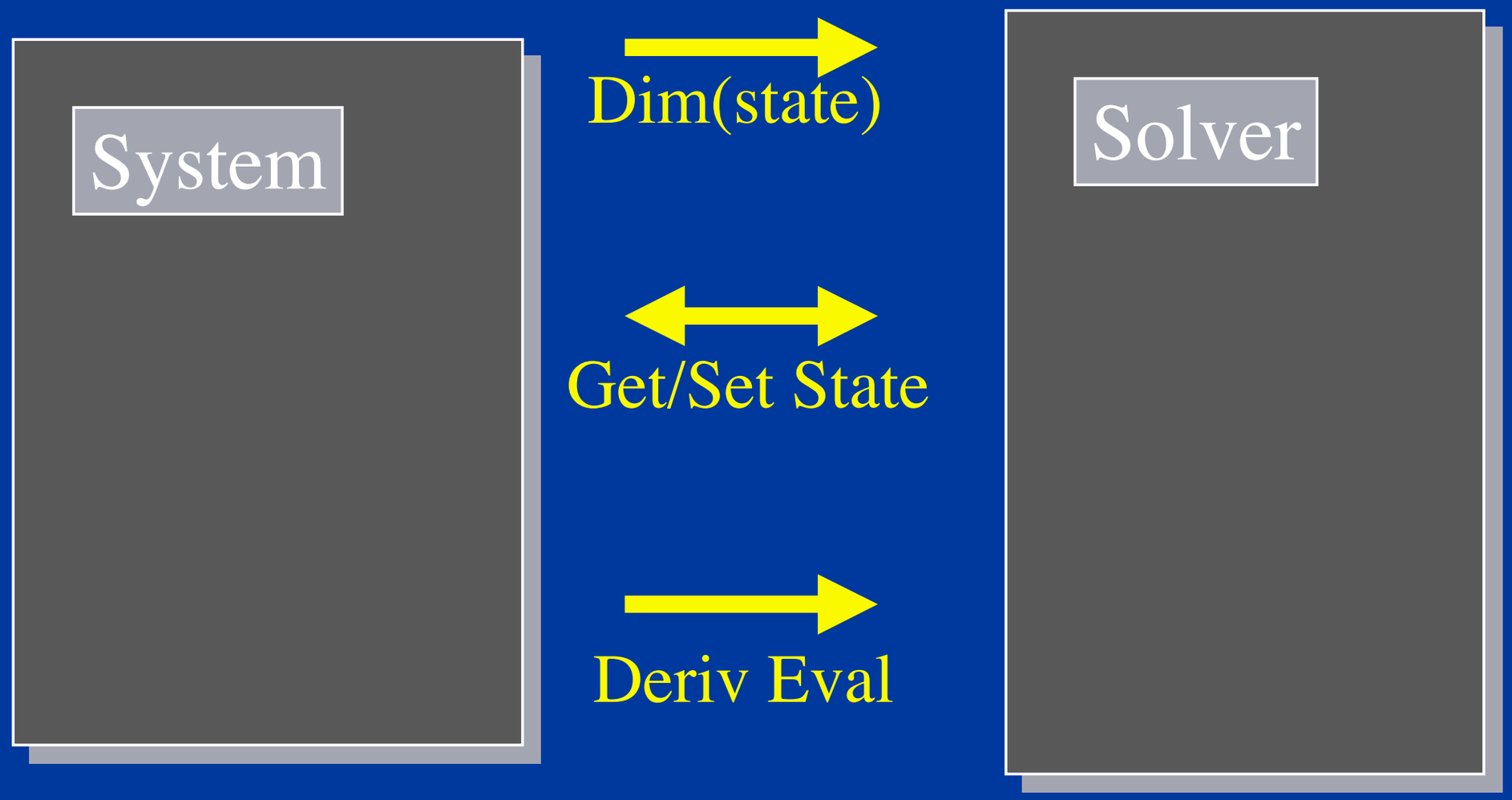
More methods…

- Euler's method is *1st Order.*
- The midpoint method is *2nd Order.*
- Just the tip of the iceberg. See *Numerical Recipes* for more.
- Helpful hints:
  - *Don't* use Euler's method (you will anyway.)
  - *Do* use adaptive step size.

Modular Implementation

- Generic operations:
  - Get dim(x)
  - Get/set x and t
  - Deriv Eval at current (x,t)
- Write solvers in terms of these.
  - Re-usable solver code.
  - Simplifies model implementation.

Solver Interface



A Code Fragment

```
void eulerStep(Sys sys, float h) {
   float t = getTime(sys);
   vector<float> x0, deltaX;

   t = getTime(sys);
   x0 = getState(sys);
   deltaX = derivEval(sys,x0, t);
   setState(sys, x0 + h*deltaX, t+h);
}
```

Question

- How could we model the solar system with differential equations?
  - What would the variables be?
  - What would the equations be (roughly)?
- What sorts of common physical phenomena are *not* well modeled by differential equations?

Student Answers

How can we model the solar system?

- each variable is a planet location + velocity:
    - 6 * n variables for n planets
- second order differential equation (define acceleration)
- need to know mass, volume, and several other variables (SOME OF CONSTANT - OR DERIVABLE)
- simplifying assumption: only depends on the sun!
- to compute all force pairs - need n^2 computation

What sorts of phenomena are not well modeled?

- things that travel very fast
    - light (travels "instantaneously")
    - electricity
- phenomena which happen much faster than "timestep"
    - collisions (DISCONTINUITIES!)
- free will
    - human motion (!!)
- tearing, molecular properties of a material