15-859N — Spectral Graph Theory and Numerical Linear Algebra. — Fall 2013

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Assignment 5
Due date: Friday Dec 6, 2013

1. (10+10 points) Draining Overflows

   In class, we briefly talked about how to deal with overflow by *draining*. Let $G = (V, E)$ be a directed graph (which corresponds to a flow that might have some overflow). It does not have a self-loop and for any two vertices u, v, there is at most one edge ($u \to v$, $v \to u$, or nothing). For simplicity, assume G is unweighted. For any vertex v, let $\mathsf{indeg}(v)$ be the number of edges going into v, and $\mathsf{outdeg}(v)$ be the number of edges going out of v. Suppose we have three special vertices s, t, d. They satisfy

   - Flow conservation: for any vertex $v \neq s, t$, $\mathsf{indeg}(v) = \mathsf{outdeg}(v)$.
   - Source-Sink property: $\mathsf{indeg}(s) = \mathsf{outdeg}(t) = 0$.

   Let $o(E) := \mathsf{indeg}(d) = \mathsf{outdeg}(d)$, and $f(E) := \mathsf{outdeg}(s) = \mathsf{indeg}(t) \geq o(E)$.

   1. Prove that there exists subset $E' \subseteq E$ such that the graph $G' = (V, E')$ satisfies the above two properties and $\mathsf{indeg}(d) = \mathsf{outdeg}(d) = 0$, with $f(E') \geq f(E) - o(E)$. Show how to find it in $O(m + n)$ time.
   2. Now, let G be *weighted*; there is a weight $w(e) \in \mathbb{R}^+$ associated with each edge e. $\mathsf{indeg}(v)$ (resp. $\mathsf{outdeg}(v)$) is now defined to be the sum of the weights of the edges going into (resp. out of) v. Flow conservation and Source-Sink property is well-defined in G. Prove that there exists $E' \subseteq E$ and $w' : E' \to \mathbb{R}^+$ such that $G' = (V, E')$ with weight w' satisfies these properties and $\mathsf{indeg}(d) = \mathsf{outdeg}(d) = 0$, with $f(E') \geq f(E) - o(E)$ (Computing them efficiently uses a sophisticated data structure, which is beyond the scope of this course)

2. (25 points) Random walk to round fractional flows

   Let $G = (V, E)$ be a unit-weighted undirected graph, and $f : E \mapsto [-1, 1]$ be a *st* fractional flow with value F (as usual, we give an arbitrary orientation to an undirected edge). We want to quickly find an integral *st* flow with value $\lfloor F \rfloor$. The following parts walk you through the algorithm in Lee-Rao-Srivastava paper.

   1. Explain how to add a new node $\tilde{s}$ so that the flow is a circulations with a flow of F passing through $\tilde{s}$ and explain how to get a Markov chain (random walk) on this new graph where the probability of leaving each vertex is proportional to the flow on the edge.

2. Determine the stationary distribution for the Markov chain and explain why it is irreducible. Read chapter one of "Markov Chains and Mixing Times" in the Related work directory and explain how the stationary is related to the expected return time.
3. Explain how to find an augmenting path P of weight one in $\tilde{G}$ and the expected time to find it.
4. Given the path P describe the construction of a residual graph and flow $F - 1$ and determine the stationary distribution for this graph.
5. Use these ideas to find an integral flow of size $\lfloor F \rfloor$ in expected time $O(m \log n)$.
6. Can you find a flow of size $\lceil F \rceil$ in expected time $O(m \log n)$?
7. Suppose that we have a fractional match for a bipartite graph for weight F. That is we have an assignment of fractional values to the edges such at the total weight on the edges of a vertex is at most one. Show how to find a matching of size $\lfloor F \rfloor$ in expected $O(m + n \log n)$ time where the weights on the edges at each vertex are given in sorted order.

3. (15 points) Number of spanning trees in terms of eigenvalues

   Let M be a symmetric $n \times n$ matrix such that all row and column sums are 0 (but it does not have to be a positive semidefnitie). One of the eigenvalues of M is $\lambda_1 = 0$; let $\lambda_2, ..., \lambda_n$ denote the other eigenvalues. Show that all principal $(n-1) \times (n-1)$ submtracies of M have the same determinant, and this value is equal to the product $\frac{1}{n}\lambda_2\lambda_3...\lambda_n$ (Hint: Consider the characteristic polynomial $\mathsf{det}(xI - M)$, and calculate one of the coefficients in two ways).

4. (15 points) The s-t Minimum cut problem.

   Suppose that $G = (V, E, w)$ is weighted undirected graph and the $s \neq t$ are two vertices of G. The Minimum cut problem is finding a minimum weight subset of edges $C \subset E$ such that removing them disconnects s from t. Consider the following l_1 minimization problem.

$$LP = \min_{x \in \mathbb{R}^n} \sum_{ij \in E} |x_i - x_j| \quad \text{where} \quad x_s = 0 \quad \text{and} \quad x_t = 1$$

   Show that $minCut = LP$.