1. (3+3+4+5 points) Construction of Low Stretch Spanning Trees

This problem tries to show an algorithm for finding a spanning tree with $O(m^{1/2} \log n)$ average stretch in an unweighted graph with $m$ edges.

1. Suppose we have a BFS tree of the graph starting at some vertex. Let $E_l$ be the number of edges going from vertices at distance $l$ to $l+1$ and $B_l$ be the number of edges between vertices at distances $[0,l]$. Show that there exist $l \leq O(m^{1/2} \log n)$ such that $E_l \leq \frac{m-1}{2}B_l$.

2. Show that in an unweighted graph, we can decompose it into pieces of radius $O(m^{1/2} \log n)$ so that at most $m^{1/2}$ edges are between the pieces.

3. Give an algorithm that generates a spanning tree with average stretch $O(m^{1/2} \log n)$.

4. Generalize it to an algorithm that for any fixed $k$, generates a spanning tree with average stretch $O(m^{1/k} \log n)$ (the constant hidden in $O$ may depend on $k$).

2. (10+5 points) A Bad Example for Spectral Partitioning

Define threshold spectral partitioning of a possibly weighted graph $G = (V,E)$ to be the vertex partition one gets by; 1) Finding the eigenvector $x$ for $\lambda_2$, 2) Sorting the vertices by their value in $x$, and 3) Returning the best threshold cut.

Let $P_{n^c}$, for $n$ even, be the weighted path graph on $n$ vertices where all the edges have unit weight except the middle one which has weight $\epsilon$.

Consider the Cartesian product $M_{cn}^\epsilon = P_{c\sqrt{n}} \otimes P_{\sqrt{n}}^\epsilon$

1. Show that threshold spectral partitioning on the graph $M_{cn}^\epsilon$ will generate a quotient cut of size $\Omega(1/\sqrt{n})$ for $c$ sufficiently large and $\epsilon = 1/\sqrt{n}$ while the best cut is of size $O(1/n)$.

2. How big does $c$ need to be for this to happen?

3. (8+5+7 points) Spectrum of 3-colorable Graphs

Let $G = (V,E)$ be a $d$-regular graph that is 3-colorable and such that there is a 3-coloring in which the color classes have equal size $|V|/3$. Let $A$ be the adjacency matrix and $L = I - \frac{A}{d}$ be the normalized Laplacian of $G$.

1. Prove that $L$ has at least two eigenvalues which are at least $3/2$, that is, $\lambda_{n-1} \geq 3/2$.

[Hint: Is it enough to present two linearly independent vectors with Rayleigh quotient $\geq 3/2$?]
2. Given an example in which the bound is tight.

3. Show that the converse is not true (that is, give an example of a regular graph that is not 3-colorable but such that at least two eigenvalues of the normalized Laplacian are $\geq 3/2$).

4. (10 points) Generalized Eigenvalues between Path and Cycle

In class, we used the fact that the generalized eigenvalues between $P_n$ and $C_n$ are 0 (with multiplicity 1), 1 (with multiplicity $n - 2$), and $n$ (with multiplicity 1). Prove this fact.

5. (10 points) Sampling Multi-graphs

In class, we said that given two positively weighted graphs $G$ and $H$ on the same set of vertices the $G \preceq H$ if $\forall x \ x^T L_G x \leq x^T L_H x$. If $G$ and $H$ are weighted multi-graphs then we shall say that $G \preceq H$ if $\forall x \in \mathbb{R}^n$ we have that

$$\sum_{e(i,j) \in E(G)} W_e(i,j)(x_i - x_j)^2 \leq \sum_{e(i,j) \in E(H)} W_e(i,j)(x_i - x_j)^2$$

where $e(i,j)$ is the multiedge with end points $i$ and $j$.

Suppose we now take each edge in $G$ and partition its weight amongst some multiple copies of the edge, giving a multigraph $H$.

1. How is the spectral equivalence effected?

2. If we plan on using the procedure SAMPLE from lecture where edge $e$ is sampled with frequency $p'_e$ what should be the sampling frequency for the edges of $H$ be to preserve spectral equivalence?

3. How does the two sparsified graphs for $G$ and $H$ compare when using AW?