15-859N — Spectral Graph Theory and Numerical Linear Algebra. — Fall 2013

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Assignment 4 Due date: Friday Nov 15, 2013

1. (3+3+4+5 points) Construction of Low Stretch Spanning Trees

   This problem tries to show an algorithm for finding a spanning tree with $O(m^{1/2} \log n)$ average stretch in an unweighted graph with m edges.

   1. Suppose we have a BFS tree of the graph starting at some vertex. Let E_l be the number of edges going from vertices at distance l to $l+1$ and B_l be the number of edges between vertices at distances $[0, l]$. Show that there exist $l \leq O(m^{1/2} \log n)$ such that $E_l \leq m^{-1/2} B_l$.
   2. Show that in an unweighted graph, we can decompose it into pieces of radius $O(m^{1/2} \log n)$ so that at most $m^{1/2}$ edges are between the pieces.
   3. Give an algorithm that generates a spanning tree with average stretch $O(m^{1/2} \log n)$.
   4. Generalize it to an algorithm that for any fixed k, generates a spanning tree with average stretch $O(m^{1/k} \log n)$ (the constant hidden in O may depend on k).

2. (10+5 points) A Bad Example for Spectral Partitioning

   Define **threshold spectral partitioning** of a possibly weighted graph $G = (V, E)$ to be the vertex partition one gets by; 1) Finding the eigenvector x for λ_2, 2) Sorting the vertices by their value in x, and 3) Returning the best threshold cut.

   Let P_n^ϵ, for n even, be the weighted path graph on n vertices where all the edges have unit weight except the middle one which has weight ϵ.

   Consider the Cartesian product $M_{cn}^\epsilon = P_{c\sqrt{n}} \otimes P_{\sqrt{n}}^\epsilon$

   1. Show that threshold spectral partitioning on the graph M_{cn}^ϵ will generate a quotient cut of size $\Omega(1/\sqrt{n})$ for c sufficiently large and $\epsilon = 1/\sqrt{n}$ while the best cut is of size $O(1/n)$.
   2. How big does c need to be for this to happen?

3. (8+5+7 points) Spectrum of 3-colorable Graphs

   Let $G = (V, E)$ be a d-regular graph that is 3-colorable and such that there is a 3-coloring in which the color classes have equal size $|V|/3$. Let A be the adjacency matrix and $L = I - \frac{A}{d}$ be the normalized Laplacian of G.

   1. Prove that L has at least two eigenvalues which are at least $3/2$, that is, $\lambda_{n-1} \geq 3/2$. [Hint: Is it enough to present two linearly independent vectors with Rayleigh quotient $\geq 3/2$?]

2. Given an example in which the bound is tight.
3. Show that the converse is not true (that is, give an example of a regular graph that is not 3-colorable but such that at least two eigenvalues of the normalized Laplacian are $\geq 3/2$).

4. (10 points) Generalized Eigenvalues between Path and Cycle

   In class, we used the fact that the generalized eigenvalues between P_n and C_n are 0 (with multiplicity 1), 1 (with multiplicity $n-2$), and n (with multiplicity 1). Prove this fact.

5. (10 points) Sampling Multi-graphs

   In class, we said that given two positively weighted graphs G and H on the same set of vertices the $G \preceq H$ if $\forall x \; x^T L_G x \leq x^T L_H x$. If G and H are weighted multi-graphs then we shall say that $G \preceq H$ if $\forall x \in \mathbb{R}^n$ we have that

$$\sum_{e(i,j) \in E(G)} W_e(i,j)(x_i - x_j)^2 \leq \sum_{e(i,j) \in E(H)} W_e(i,j)(x_i - x_j)^2$$

   where $e(i,j)$ is the multiedge with end points i and j.

   Suppose we now take each edge in G and partition its weight amongst some multiple copies of the edge, giving a multigraph H.

   1. How is the spectral equivalence effected?
   2. If we plan on using the procedure SAMPLE from lecture where edge e is sampled with frequency p'_e what should be the sampling frequency for the edges of H be to preserve spectral equivalence?
   3. How does the two sparsified graphs for G and H compare when using AW?