15-859N — Spectral Graph Theory and Numerical Linear Algebra. — Fall 2013

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Assignment 3 Due date: Friday October 25, 2013

1 Expander better than Ahlswede-Winter

[20 ponts] One important property of the complete graph $K_n = (V, E)$ is *expansion*. We say that a weighted graph has c-expansion if for any subset $S \subseteq V$ with $|S| \leq \frac{n}{2}$, $|E(S, \bar{S})| \geq cn|S|$ where $E(S, \bar{S})$ is the sum of the weight of edges between S and $\bar{S}$. This definition is scale-variant, but all graphs considered in this problem will have the sum of weights $\Theta(n^2)$. It is also easy to see that K_n is a $\frac{1}{2}$-expander.

1. In class, we showed that there exists a weighted graph G with $O(n \log n)$ edges such that

   $$(1 - \epsilon)K_n \preceq G \preceq (1 + \epsilon)K_n$$

   Show that G is a c-expander for some c, which may depend on ϵ.

2. We give an easy proof that expansion is achieved by a graph with only linear number of edges. Let $G(n, d)$ be a (slightly unusual) random graph such that

   - At each vertex v, we choose d vertices $u_1, ..., u_d$ independently from $V \setminus \{v\}$.
   - Add edges (v, u_i) for all $1 \leq i \leq d$.

   Note that v's degree may be greater than d if it is *chosen* by other vertices, and the number of edges is always nd.

   Show that there exists d such that $\frac{n}{d}G(n, d)$ (so that the sum of weights is exactly n^2) is $\frac{1}{4}$-expander with high probability.

   HINT: It seems that the version of Chernoff bounds given in class is not strong enough to prove that $\frac{n}{d}G(n, d)$ is an expander. Try the following version, see Wikipedia,

   Let random variables $X_1, \ldots, X_n$ be independent random variables taking on values 0 or 1. Further, assume that $\mathbf{P}(X_i = 1) = p_i$. Then, if we let $X = \sum_{i=1}^{n} X_i$ and μ be the expectation of X, for any $\delta > 0$

   $$\mathbf{P}(X > (1 + \delta)\mu) < \left(\frac{e^{\delta}}{(1 + \delta)^{(1+\delta)}} \right)^{\mu}.$$

   Besides using this version of Chernoff bounds one should consider the random variable which is the number of edges internal to some set of vertices S. This random variable will allow you to use a large value for δ in your proof.

2 Chebyshev Polynomials

[20 points]

In this problem we will develop some important identities for Chebyshev Polynomials.

1. Consider the following rotation matrix by angle θ:

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Check that A_θ^n is a rotation by $n\theta$, even if θ is a complex number.

2. We can abstract A_θ to a matrix $A = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ where $c^2 + s^2 = 1$. Show that

$$A^n = \begin{pmatrix} T_n(c) & -sQ_n(c) \\ sQ_n(c) & T_n(c) \end{pmatrix}$$

where T_n and Q_n are polynomials in c satisfying:

$$\begin{aligned} T_0(c) &= 1 \\ T_1(c) &= c \\ T_{n+1}(c) &= cT_n(c) - (1 - c^2)Q_n(c) \end{aligned}$$

and

$$\begin{aligned} Q_0(c) &= 0 \\ Q_1(c) &= 1 \\ Q_{n+1}(c) &= cQ_n(c) + T_n(c) \end{aligned}$$

3. Use these identities to show that:

$$\begin{aligned} T_{n+1}(c) &= 2cT_n(c) - T_{n-1}(c) \\ Q_{n+1}(c) &= 2cQ_n(c) - Q_{n-1}(c). \end{aligned}$$

Thus T and Q are Chebyshev Polynomials of the first and second kind respectively. Explain why all the roots of T_n and Q_n lie in the interval $[-1, +1]$ and in this interval T returns values in this interval.

4. Show how to diagonalize A.

5. Use this diagonal form to show that

$$T_n(c) = \frac{(c + \sqrt{c^2 - 1})^n + (c - \sqrt{c^2 - 1})^n}{2} = \frac{(c + \sqrt{c^2 - 1})^n + (c + \sqrt{c^2 - 1})^{-n}}{2}$$

HINT: To get the last equality use the fact that c is $\cos\theta$ and use complex numbers.

3 Eigenvalues of Cartesian Products

[15 points]

Let $G = (V, E, w)$ and $H = (V', E', w')$ be two non-negatively weighted simple graphs. Let $G \otimes H = (\bar{V}, \bar{E}, \bar{w})$ be their Cartesian product, where:

- The vertices are $\bar{V} = V \times V'$
- The edges are $\bar{E} = \{((x, x'), (y, y')) \mid [x = y \wedge (x', y') \in E'] \vee [x' = y' \wedge (x, y) \in E]\}$
- $\bar{w}((x, x'), (x, y')) = w'(x', y')$ and $\bar{w}((x, x'), (y, x')) = w(x, y)$.

1. Show that the eigenvalues of $L_{G\otimes H}$ are the direct sum of those of L_G and L_H. That is if the eigenvalues of L_G are $\{\lambda_1, \ldots, \lambda_n\}$ and those of L_H are $\{\mu_1, \ldots, \mu_m\}$ the those of $L_{G\otimes H}$ are $\{\lambda_i + \mu_j \mid 1 \le i \le n,\ 1 \le j \le m\}$
2. Show that the eigenvectors of $L_{G\otimes H}$ are the direct product of those of L_G and L_H.

4 Conjugate Directions

[15 points] Conjugate Directions is another method for solving $Ax = b$, where $A \in \mathbb{R}^{m\times m}$ is SPD. Suppose that we are given a set of search directions $\left\{d_{(0)}, ..., d_{(m-1)}\right\}$ such that $d_{(i)}^T A d_{(j)} = 0$ for all $i \neq j$ (Conjugate Gradient is just a way of efficiently choosing such $d_{(i)}$'s using gradients). Given an initial point $x_{(0)}$, we update in each iteration such that

$$x_{(i+1)} \leftarrow x_{(i)} + \frac{d_{(i)}^T r_{(i)}}{d_{(i)}^T A d_{(i)}} d_{(i)}$$

where $r_{(i)} = b - Ax_{(i)}$. It means that in step (i), we find $x_{(i+1)}$ such that $r_{(i+1)} \perp d_{(i)}$.

Prove that this method converges in $i \le m - 1$ steps.

Hint: Write the initial error $e_{(0)}$ as a linear combination of $d_{(i)}$'s and show that ith step annihilates ith component.