15-859N — Spectral Graph Theory and Numerical Linear Algebra. — Fall 2013

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Assignment 2 Due date: Monday October 7, 2013

Please turn in each problem on a separate sheet of paper for grading purposes. Each sheet should include your name.

1 Random Walks and Trolls

[10 points]

Consider the following variant of the random walk problem: you still proceed around the vertices in a random walk in an undirected graph with Laplacian L, but there are trolls on each of the vertices. Specifically vertex u contains x_u trolls. Given the vector $\mathbf{u}$ of number of trolls on all vertices, give an expression for the expected number of trolls that you encounter when walking from s to t. (you may use $\mathbf{e}_u$ to denote the vector that's 1 at u and 0 everywhere else.

2 Another Proof of the Perron-Frobenius Theorem

[5 + 10 + 5 = 20 points]

We give another, more direct proof of a slightly weaker version of the Perron-Frobenius theorem. We weaken the theorem in two ways. First we assume the matrix is a transition matrix, namely, the columns sums are all one. Second, instead of showing that the stationary distribution converges, we show that the average of all distribution converges. That is, let $x^{(1)}$ be any initial probability vector and define:

$$x^{(t+1)} = Ax^{(t)}$$

and the running average

$$y^{(t)} = \frac{1}{t} \sum_{1 \leq t' \leq t} x^{(t')}$$

where the column sums of A are all one, $A \geq 0$, and A is irreducible, that is, if we represent the A as a directed graph (there is an edge $i \to j$ if $A_{j,i} > 0$), then this graph is strongly connected.

1. Let $z^{(t)} = Ay^{(t)} - y^{(t)}$. Show that $z^{(t)} = \frac{1}{t}(x^{(t+1)} - x^{(1)})$
2. Show that $y^{(t)} = Bz^{(t)} + c$ where B is a linear transformation and c is a fixed vector, both depending only on A.
   Hint:
   1) Show that the left null space of $A - I$ is generated by the all ones verctor.
   2) Let B' be the $(n+1) \times n$ matrix obtained by appending $\mathbf{1}$ to $A - I$. Determine the left null space of B'.
   3) Show how to get B by dropping a row and colmun of B'
3. Conclude that $y^{(t)}$ converges to some limit.

3 Bounding Separators Sizes using Path Embeddings

[10 points]

Given a graph $G = (V, E)$ an **edge separator** is a set of edges

$$E(A, B) = \{(uv) \in E \quad | \quad u \in A \text{ and } v \in B\}$$

where A B form a partition of the vertices V and $|A|, |B| \geq |V|/3$. The size of the separator is $|E(A, B)|$.

Show that any edge separator for the 2 dimensional mesh with n nodes has size $\Omega(\sqrt{n})$ using path embeddings.
Hint: Give a lower bound for separators for the complete graph and use this to bound the mesh.

4 λ_2 of Balanced Binary Tree

[15 points] Let T_n be a rooted balanced binary tree with n vertices (i.e. the depth of the left and right subtrees of every node differ by 1 or less). Show that λ_2 of its Laplacian is $\Theta(1/n)$.

5 Another Property of Effective Resistance

[10 points] In the class, we proved that $G_{1,n} \preceq \alpha G$ implies $\mathsf{ER}_{1,n} \leq \alpha$. Prove that the converse is also true.