15-859N — Spectral Graph Theory and Numerical Linear Algebra. — Fall 2013

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Assignment 1 Due date: Monday September 23, 2013

Please turn in each problem on a separate sheet of paper for grading purposes. Each sheet should include your name.

1 Resistors in Series

[10 points]

Consider a set on n resistors $R_1, \ldots, R_n$ in series. Our goal is to show that the effective resistance between the two ends is $R = R_1 + \cdots + R_n$.

1. Show resistances in series sum using a current argument.
2. Show resistances in series sum using a voltage argument and Cauchy-Schwartz.

   Hint1: Show that for all voltage settings $V_0, \ldots, V_n$ the power will be at least $(V_0 - V_n)^2/R$.

   Hint2: Rewrite the power using the variables $\Delta_i = V_{i-1} - V_i$.
3. Redo the problem for conductors in parallel. The two methods should be reversed.

2 Resistance as a Metric

[10 points]

Show that given a graph of conductors $G = (V, E, c)$ the effective resistance between pairs of vertices form a metric space over V. That is, if we let $R(u, v)$ denote the effective resistance between u and v, show:

- $R(u, v) \geq 0$.
- $R(u, v) = 0$ if and only if $u = v$
- $R(u, v) = R(v, u)$.
- $R(u, v) + R(v, w) \geq R(u, w)$.

Hint: To show the triangle inequality I think a current argument may be the simplest.

3 Effective Resistance

[5+15=20 points]

Let M_n be the unit weight square mesh graph on n nodes, in this problem we try to obtain asymptotically tight bound on the effective resistance between the opposite corners.

1. Show the effective resistance between the opposite corners of M_n $\Omega(\log n)$.

   Hint: use the Rayleigh Monotonicity Theorem.

2. Show that the effective resistance between the opposite corners is $O(\log n)$.

   Hint: One interesting method is to use Polya's Urn Process, see wikipeda.

4 Randomized s-t Connectivity

[5+10+5=20 points] Suppose we're given an undirected graph $G = (V, E)$ and two nodes s and t in G, and we want to determine if there is a path connecting s and t. This is easily accomplished in $O(|V| + |E|)$ using DFS or BFS. The space usage of such algorithms, however, is $O(|V|)$.

In this problem, we will consider a randomized algorithm that solves the s-t connectivity problem using only $O(\log |V|)$ space. Here is a very simple algorithm which we will analyze.

Step 1: Start a random walk from s.
Step 2: If we reach t within $4|V|^3$ steps, return "`CONNECTED`." Otherwise, return "`NO`."

Assume that G is not bipartite. Your proof doesn't have to match the constants below exactly; they just have to be in the same ballpark.

1. Prove that for any edge $(u, v) \in E$, $C_{u,v} \leq 2|E|$, where $C_{u,v}$ is the commute time between u and v. *(Hint: Rayleigh's Monotonicity Principle.)*

2. Let $\mathcal{C}(G; v)$ be the expected length of a walk starting at u and ending when it has visited every node of G at least once. The *cover time* of G, denoted $\mathcal{C}(G)$, is defined to be

$$\mathcal{C}(G) := \max_{v \in V(G)} \mathcal{C}(G; v).$$

   Show that the cover time of $G = (V, E)$ is upper bounded by $2|V||E|$. *(Hint: Construct an Euler's tour on a spanning tree.)*

3. Conclude that the algorithm above is correct with probability at least $3/4$. *(Hint: Markov's inequality.)*