Random Walks on Graphs

Graph \( G = (V,E,w) \) (possibly directed)
\( w: E \rightarrow \mathbb{R}^+ \)

\[
W_i = w(V_i) = \sum_{(i,j) \in E} W_{ij} \quad P_{ij} = \frac{W_{ij}}{W_i}
\]

Random walk on \( G \)

Suppose, at a given time, we are at \( V_i \in V \).
We move to \( V_j \) with probability \( P_{ij} \).

Example:
\( V = \) all permutations of a deck of cards
\( P_{ij} = \) prop of going from perm \( i \) to perm \( j \) in one shuffle.

? Why do professional players play from a deck after 5 shuffles?
Important Parameters

Access time or Hitting time
\[ H_{ij} = \text{Expected time to visit } j \text{ starting at } i \]

Commute Time
\[ K(i,j) = H(i,j) + H(i,i) \]

Cover Time
Expected time to visit all nodes
max over all starting nodes

Mixing Rate (to do)
Random walks—the Symmetric Case

Do a random walk on a network of conductors!

Input: $G = (V, E, C)$  $C_{ij} = C_{ji}$  $a, b \text{ ev}$

Consider a random walk starting at $x$ and ending at $b$.

Def $h_x = \text{prob we visit } a \text{ before visiting } b$.

$a \neq b$ starting from $x_0$.

\[ h_a = 1 \quad h_b = 0 \]

\[ h_x > \frac{1}{2} \quad \text{why?} \]
\[ h_a = 1 \land h_b = 0 \]

Suppose \( x \neq a, b \)

Claim \( h_x = \sum_y P_{xy} h_y \)

\[ P_{xy} \geq 0 \land \sum_y P_{xy} = 1 \]

\( h_x \) is a convex combination of its neighbors!

\( h \) is harmonic with boundary \( a, b \).

Let's solve the electrical prob

\( V_a = 1 \land V_b = 0 \text{ and } x \neq a, b \) float.

\[ x \neq a, b \Rightarrow V_x = \sum_y \frac{C_{xy}}{C_x} V_y \text{ but } \frac{C_{xy}}{C_x} = P_{xy} \]

\[ \Rightarrow h = V \]

Thm \( V_a = 1 \land V_b = 0 \) then \( V_x = \text{prob visit } a \text{ before } b \).
h_c = 3/4

What does it mean (in random walks)?
If we set \( V_{a_1} = V_{a_2} = 1 \) & \( V_{b_1} = V_{b_2} = 0 \)
\( X \neq a_1, a_2, b_1, b_2 \) float?
Interpretation of Current

Assume \( G = (V, E, C) \) \( a, b \) eV

Consider 1 unit of current flow from \( a \) to \( b \).

Say \( i \)

What does \( i_{xy} \) correspond to in random walks?

Thm \( i_{xy} = \text{Expected net \# of traversals of } E_{xy} \)
in random walk from \( a \) to \( b \).

pf Slides 7, 8, 8A
Let's start with:

\[ U_x = \text{Expected number of visits to } X \text{ before reaching } b \text{ starting at } a. \]

\[ U_b = 0 \quad x \neq b \]

\[ U_x = \sum_y U_y \frac{P_{yx}}{C_x} \quad \text{note } \sum_y P_{yx} \neq 1 \]

Recall \[ C_x = \sum_y C_{xy} \]

\[ \text{note } C_x P_{xy} = C_x \left( \frac{C_{xy}}{C_x} \right) = C_{xy} = C_y \left( \frac{C_{yx}}{C_y} \right) = C_y P_{yx} \]

\[ U_x = \sum_y U_y \frac{C_y P_{yx}}{C_x} = \sum_y U_y \left( \frac{P_{xy} C_x}{C_y} \right) \]

\[ \left( \frac{U_x}{C_x} \right) = \sum_y P_{xy} \left( \frac{U_y}{C_y} \right) \]
Let \( V_x = \left( \frac{U_x}{C_x} \right) \) then

\[
V_x = \sum_y P_{xy} V_y \quad V_y \text{ is harmonic}
\]

What is the boundary? \( V_b = 0 \)

Suppose we knew \( U_a \) \( V_a = U_a / C_a \)

\( V \) is a voltage where \( V_b = 0 \) & \( V_a = U_a / C_a \)

Let \( i_{xy} \) be its current

\[
\dot{i}_{xy} = (V_x - V_y) C_{xy} = \left( \frac{U_x}{C_x} - \frac{U_y}{C_y} \right) C_{xy}
\]

\[
= U_x \left( \frac{C_{xy}}{C_x} \right) - U_y \left( \frac{C_{yx}}{C_y} \right) = U_x P_{xy} - U_y P_{yx}
\]
\[ U_x P_{xy} = \text{expected # of traversals from } x \text{ to } y \]

\[ U_y P_{yx} = "y \text{ to } x" \]

\[ J_{xy} = \text{expected # net } xy \text{ traversals} \]

**What is net current flow from } a \text{ to } b ?**

\[ \sum_{Y} J_{xy} \]

This must be 1

This proves Thm
What is the algorithm?

$a=1 \& b=n$

\[
L(V) = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}
\]

Solve

\[
L(V) = \begin{pmatrix}
0 \\
0 \\
\vdots \\
-1
\end{pmatrix}
\]

set \( V' = V - v_n \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \)

\[
V_1 = \begin{pmatrix}
U_1 \\
C_1
\end{pmatrix} \quad U_1 = V_1 C_1
\]

we have found \( U_a \)


How to compute hitting Time

\[ \text{Def: } h(x,b) = \text{expected time to reach } b \text{ from } x \]
\[ h_x = h(x,b) \text{ if fixed} \]

Let's write a recurrence
\[ h_b = 0 \quad x \neq b \quad (\star) \]
\[ h_x = 1 + \sum \limits_y h_y P_{xy} \]

How do we solve (\star)?

Let's think of \( h_x \) as a voltage \( V_x \)
\[ V_b = 0 \quad V_x = 1 + \sum \limits_y \frac{C_{xy}}{C_x} V_y \]
\[ C_x V_x = C_x + \sum_{y} C_{xy} V_y \]

\[ C_x V_x - \sum_{y} C_{xy} V_y = C_x \]

\begin{align*}
\text{Graph Laplacian} & & \text{residual current} \\
\end{align*}

\[ L V = \begin{pmatrix} C_1 \\ \vdots \\ C_{n-1} \end{pmatrix} \]

\[ V_n = 0 \]

\[ b = V_n \]

by conservation of flow

\[ \delta = C_n - C \]

Alg: for hitting time

solve \[ L V = \begin{pmatrix} C_1 \\ \vdots \\ C_{n-1} \end{pmatrix} \]

\[ V_n = 0 \]

\[ V_n = C_{n-1} \]

return \[ V_x \]
What about commute time?

\( a = V_1 \) \& \( b = V_n \)

Solution 1

solve \( LV^b = \begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \\ c_n - c \end{pmatrix} \) \[ LV^a = \begin{pmatrix} c_1 - c \\ \vdots \\ c_n \end{pmatrix} \]

\( h(1,n) = V_1^n - V_n^b \)

\( h(n,1) = V_n^a - V_1^a \)

\( V = V^b - V^a \)

\( C(1,n) = (V^b - V^a)_1 - (V^b - V^a)_n = V_1 - V_n \)

Solution 2

\( L(V^b - V^a) = LV^b - LV^a = \begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \\ c_n - c \end{pmatrix} - \begin{pmatrix} c_1 - c \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_0 \end{pmatrix} = C \)
solve $LV = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

return $C(V_i - V_n)$ but $(V_i - V_n) = \mathbb{E}R_{in}$

Thm $C(a, b) = \mathbb{E}R_{ab} \cdot C$