Lecture 7: Matchings & Linear Programs

- weighted matching algorithms
  - via LPs
    - LP refresher
    - Perfect Matching Polytope
      - bipartite
      - non-bipartite

\[
\begin{align*}
\text{min-weight perfect matchings} \\
\text{max-wt match}\uparrow
\end{align*}
\]

\[\text{We} \quad G = (U, E) \quad \text{and} \quad G = (L, R, E)\]

Max Cardinality
\[
\begin{cases}
\text{Berge's Thm} & O(mn) \\
\text{König's Thm} & O(mn) \\
\text{Blossom Alg} & O(mn^2)
\end{cases}
\]
LPs \( \mathbb{R}^d \)

- \( \mathbb{R}^d \): \( \{ x : a \cdot x \geq b \} \) halfspace
- polyhedron: intersection of finitely many halfspaces

**K**: polytope - bounded polyhedron

**K** convex: \( \forall x, y \in K, \forall \lambda \in [0, 1] \) \( \lambda x + (1-\lambda)y \in K \)

Linear Program: \( \min \; \xi \cdot x \mid x \in K \)

\( \min \; c \cdot x \)

\( \text{st} \) \( a_1 \cdot x \geq b_1, \; a_2 \cdot x \geq b_2, \ldots, \; a_m \cdot x \geq b_m \)

\( \text{Max} \; x_1 + x_2 \geq 3 \)
LP: \[ \min \frac{c \cdot x}{x} \]
\[ \text{st } A x \geq b \]
\[ \exists x \]

Polyhedron: \[ K \text{ feasible region} \]

\[ \min x_1 + x_2 \]
\[ x_1 + x_2 \geq 3 \]

Extreme point of \( K \):
\[ x \in K \text{ st cannot be written as } \]
\[ x = \lambda y + (1-\lambda) z \]
\[ \lambda \in [0,1] \]
\[ y, z \in K \]

\[ y \neq z \] for \( y, z \neq x \)
(2) vertex of $K \in \mathbb{R}^d$

$z \in K$ is a vertex if

$\exists C \in \mathbb{R}^d$ for which $z$ is the unique minimizer in $K$

$\forall y \neq z \quad C^Ty > C^Tz$

(3) basic feasible solution of $K \subseteq \mathbb{R}^d$ (bfs)

$x$ is bfs of $K$ if $z \in K$ and

$\exists$ d linearly independent defining constraints of $K$ s.t.

$x$ is tight for those constraints

satisfies equality
Theorem: \( K \) be a polyhedron, \( x \) is extreme pt.
\[ \Rightarrow x \text{ is a vertex} \]
\[ \Rightarrow x \text{ is bfs.} \checkmark \]

Thm. \( K \) be a polytope,
\[ \text{LP} = \min \{ c^T x | x \in K \} \]
has an optimal pt \( x^* \) which is a vertex of \( K \).

"Thm." given an LP can solve it in polynomial time.
\[ \rightarrow \text{return a bfs of } K \]

Ellipsoid
Interior Pt Methods
Convex Hull:

$S \subseteq \mathbb{R}^d$

$\text{CH}(S)$

Fact: $K = \text{CH}($extreme pts of $K$)$

Perfect Matchings

$G = (L, R, E)$

$M \subseteq E$

$\chi_M \in \{0, 1\}^E$

$M = \{\chi_M : M \text{ is a perfect matching in } G\}$

$C_{PM}(G) = \text{CH}(M)$

Minimize $\sum c_i x_i$ subject to $x \in C_{PM}(G)$ and all PMS are corners of polytope
"Compact" way to write the PM polytope

\[ K_{PM}(G) = \{ x \in \mathbb{R}^{1 \times E} \mid \begin{align*}
\sum_{i \in E} x_{ij} &= 1 \quad \forall i \in E, \\
\sum_{j \in E} x_{ij} &= 1 \quad \forall j \in E, \\
1 \geq x_{ij} &\geq 0
\end{align*} \} \]

Thm: for bipartite \( G \), \( K_{PM}(G) = C_{PM}(G) \)

Fact: for non-bipartite \( G \), not true \( K_{PM}(G) \not\subseteq C_{PM}(G) \)
(c ∈ K) ∀M, \chi_M \in K \Rightarrow C \subseteq K

(K ∈ C) \forall x^* \in K, x^* \in C

\Rightarrow \chi^* = \chi_M \text{ for some } P, M.

Proof:

\text{Support}(x^*) = \{e \in E \mid x^*_e > 0\}

\Rightarrow \text{Support has no cycle.}

\chi = \frac{1}{2} \chi + \frac{1}{2} \chi

\Rightarrow \chi \text{ contradicts } x^* \text{ in extreme pt.}

\Rightarrow \text{Support has no cycle.} \Rightarrow \text{Support}(x^*) \text{ is a forest.}
\[ CC = K \]
\[ X = c \cap C \]
\[ \Rightarrow x^* \subseteq \chi_M \text{ for some } p_M \]
\[ p_{2}: \text{every \ line \ of \ } K \text{ \ is \ a \ matching \ (perfect)} \]

\[ k = \text{free } K \text{ \ if \ } \sum_{x \in K} x = 1 \]

\[ x = 0 \Rightarrow x_0 \approx 0 \]

\[ \sum_{x \in K} x = 1 \text{ \ for \ } \forall \ v \in K \text{ \ such \ that} \ x \in K \]

\[ x = 0 \Rightarrow x_0 \approx 0 \]

\[ \sum_{x \in K} x = 1 \text{ \ for \ } \forall \ v \in K \text{ \ such \ that} \ x \in K \]
\( Ax = b \)
\[ x_i = \frac{\det(A|b|_i)}{\det(A)} \]

\( C x^* = 1 \)
\[ x_e^* = \frac{\det(C[1] e)}{\det(C)} \]
\[ \det(C) \in \mathbb{Z} + i\mathbb{Z} \]

\( C \) is non-singular

Claim

\( \min \{ c : x \in K_{PM} \} \)