Lecture 14: Experts and Bandits

- The Mistake-Bound Model (online)
- Algorithms for Prediction
- Randomization
- Bandits?
$N$ experts

Each day/round $(t)$

- Each expert gives prediction
- Algorithm makes prediction
- World makes an actual outcome
- If $(o_t \neq a_t)$ -> mistake

Want: # mistakes small.

$E \subseteq U^N$

$U = \{y, N\}$

<table>
<thead>
<tr>
<th>day</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>y</td>
<td>y</td>
<td>N</td>
<td>y</td>
<td>y</td>
</tr>
<tr>
<td>predict y</td>
<td>predict N</td>
<td>actual y</td>
<td>actual N</td>
<td></td>
<td></td>
</tr>
<tr>
<td>mistake</td>
<td>mistake</td>
<td></td>
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</table>
Q: Sps is a perfect expert, get ≤ N-1 mistakes?

Q: Is the best expert makes ≤ m* mistakes. What can we do?

Q: Want an algo

best expert \( \hat{e} \) has made \( m^*(t) \) errors

I want to make \( \leq m^*(T) \) errors

\[ m(t) \leq \log_2 N \]
Claim: I algo. that # males ≤ \((m^*(t)+1)(\log_2 N + 1)\) mistakes.

Proof:
Use same algo. as before.

\(m^*(t) = 0\)

\(\Rightarrow\) Error means there, all balls.

\(\text{# epochs ≤ } (m^*(t)+1)\)
\(\text{# mistakes/epoch ≤ } (\log_2 N + 1)\)

\((H-E) ≤ m^*(t) + \frac{\log N}{\varepsilon}\)
Weighted Majority (WM) / Multiplicative Weight (MW) Littlestone/Warmuth

\[
\Phi^t = \sum_{i=1}^{N} w_i^t
\]

Each expert has weight \( w^t \) and \( w_i^t \leq 1 \).

At each day \( t \):
1. Predict weighted majority \( \Phi^t \).
2. See \( \Phi^t \), if expert \( i \) is incorrect, \( w_i^{t+1} = \frac{1}{2} w_i^t \).
3. Else, \( w_i^{t+1} = w_i^t \).

**Thm:** # mistakes by WM by time \( T \) is at most \( 2.41 (m^*(T) + \log N) \).

Proof: \( \Phi^t = N \), \( \Phi^{t+1} \leq \Phi^t \) sps Alg errs: \( \Phi^{t+1} \leq \Phi^t (3/4) \).

\[
\phi^{\text{min}} = \frac{1}{2} \leq \Phi^{t+1} \leq N \left( \frac{3}{4} \right)
\]

\[
M = \frac{m^* \log 2 + 1 + \log N}{\log (3/4)}
\]
Thm. \( MW(\frac{1}{2}) \) gives \( M \leq \left(m^* + \log N\right) / 2.4 \)

\( MW(1-\epsilon) \) gives \( M \leq 2(1+\epsilon) m^* + O\left(\frac{\log N}{\epsilon}\right) \)

 determmistic

Prop. # def alg A, Input m which \( M \geq 2 m^* \)

Pf: 2 experts. \( E \): yes. \( E_2 \): no.

\[
\begin{array}{c|c|c|c}
\text{a} & \text{Y} & \text{N} & \text{Y} \\
\text{b} & \text{N} & \text{N} & \text{N} \\
\end{array}
\]

\( M = \# \text{days} \)

\( \exists \text{ expert } \# \text{ mistakes} \leq \frac{\# \text{days}}{2} \)
Randomized WM (RWM / RMW)

- \( W_i^1 = 1 \quad \forall i \in \mathbb{N} \)
- on day \( t \), [predict \( Y \) wp. \( \frac{\sum_i W_i^t}{\sum_i W_i^t} \), \( N \) otherwise]

Upon seeing \( o_t \), update as before:

\[
\begin{cases}
W_i^{t+1} = W_i^t (1 - \varepsilon) & \text{if } i \, \text{wrong} \\
W_i^{t+1} = W_i^t & \text{if } i \, \text{right}
\end{cases}
\]

Thm: \( E[\# \text{ mistakes}] \leq (1 + \varepsilon) M^*(T) + O \left( \frac{\log N}{\varepsilon^2} \right) \)

**Pf:** \( F^t = \text{Prob of making error at } t = \frac{\sum_i W_i^t \text{ mistake}}{\sum_i W_i^t} \)

\[
\Phi^t = \Phi^t [1 - F_t] + F_t (1 - \varepsilon) = \Phi^t (1 - \varepsilon F_t) = \Phi^t \prod_{s \leq t} (1 - \varepsilon F_s)
\]

Final Obs: \( E[M] = \sum_{s \leq t} F_s \)

\( \in \mathbb{N}, e^{-\varepsilon \sum_{s \leq t} F_s} \leq N . e^{-\varepsilon . E[M]} \)
Dot Product Game (Def Game)

At each time $t$ (simultaneously):

- also produces $p_t \in \{0, 1\}^N$, $\sum_i p_i = 1$ ? $\Delta N$

- world produces $l_t \in [-1, 1]^N$

Loss of agent $a$ at day $t = \langle l_t, p_t \rangle$

$E[\text{loss in proving game}] = \sum_i p_i l_i$
Algorithm [Hedge / variant of MW, RMW]

- \( w_i^t = 1 + t \)

- at each time

\[
\begin{align*}
\phi_t & = \frac{w_t^\top}{\sum_i w_i^t} \\
\phi_t^i & = e^{-\phi_t \cdot i} \\
\forall \phi \in \Phi & \ni \sum_i \phi_i^t \leq \frac{1}{T} \sum_{t=1}^T \langle e_t, \phi_t \rangle + \frac{\ln N}{e^T} + \frac{\Theta}{T}
\end{align*}
\]

Thm: Fix \( \frac{e \ln N}{e} \). For any sequence of \( e_1, e_2, \ldots, e_T \in [-1, 1]^N \), the Hedge algorithm with average losses

\[
\frac{1}{T} \sum_{t=1}^T \langle e_t, \phi_t \rangle \leq \min_{\phi \in \Phi} \frac{1}{T} \sum_{t=1}^T \langle e_t, \phi \rangle + \frac{\Theta}{T} + \frac{\ln N}{e^T}
\]

Vainly regret

\( \forall t \in [T] \ni e_t = \ln N/e \Rightarrow 2 \sqrt{T \ln N} \)