Graduate AI

Lecture 3: Search II

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A* IS OPTIMALLY EFFICIENT

• We say that node $x$ is surely expanded by A* tree search if $f(x) < f(t^*)$, where $t^*$ is the optimal goal.

• Theorem [Dechter and Pearl 1985]: Any tree search algorithm that is optimal given a consistent heuristic will expand, whenever the heuristic is consistent, all nodes surely expanded by A*.
Proof of Theorem

- Let \( I \) be an instance with graph \( G \) and consistent heuristic \( h \)
- Assume node \( x \) is surely expanded by A*
- Denote \( f(x) < f(t^*) = C^* \)
- Let \( B \) be an optimal algorithm that does not expand \( x \)
**Proof of Theorem**

- Create $G'$ by adding a new goal $t$, and an edge $(x, t)$ with cost $h(x)$
- $h'$ is the same as $h$, and $h'(t) = 0$
- **Lemma:** $h'$ is consistent
  - Clearly true on pairs that do not include $t$
  - For pairs $(y, t)$,
    $$h'(y) = h(y) \leq c(y, x) + h(x) = c(y, t) = c(y, t) + h'(t)$$
Proof of Theorem

- On the new instance $I'$ defined by $G'$ and $h'$, A* will find the goal $t$ with cost
  \[ g(t) + h'(t) = g(x) + h(x) < C^* \]
- Because $B$ does not expand $x$, $I'$ looks identical to $I$, and $B$ will find a solution of cost $C^*$
- This is a contradiction to the assumption that $B$ is optimal whenever the heuristic is consistent $\blacksquare$

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Why does the proof fail if we replace “consistent” with “admissible”?
A* is optimally efficient

- In fact, the theorem is false when the heuristic is only admissible
- In the example on the right, algorithm B will find the optimal solution while expanding fewer nodes than A*

Alg B: Conduct exhaustive search except for expanding a; then expand a only if it has the potential to sprout cheaper solution
Application: Motion Planning

"His path-planning may be sub-optimal, but it's got flair."
Motion planning

- Navigating between two points while avoiding obstacles
- A first approach: define a discrete grid
- Mark cells that intersect obstacles as blocked
- Find path through centers of remaining cells
Is this approach optimal? Complete?
Cell decomposition

- Distinguish between
  - Cells that are contained in obstacles
  - Cells that intersect obstacles
- If no path found, subdivide the mixed cells
Is it complete now?

- An algorithm is **resolution complete** when:
  a. If a path exists, it finds it in finite time
  b. If a path does not exist, it returns in finite time
- Assume obstacles are closed sets, so free space is an open set
- **Poll 1:** Cell decomposition satisfies:
  1. a but not b
  2. b but not a
  3. Both a and b
  4. Neither a nor b
Cell decomposition

Shortest paths through cell centers

Shortest path
**Solution 1: A* Smoothing**

- Allows connection to further states than neighbors on the grid
- Key observation:
  - If $x_1, \ldots, x_n$ is valid path
  - And $x_k$ is visible from $x_j$
  - Then $x_1, \ldots, x_j, x_k, \ldots, x_n$ is a valid path
SMOOTHING WORKS!

--- A shortest path through cell centers
----- Shortest path
SMOOTHING DOESN’T WORK!

A shortest path through cell centers

Shortest path
**Solution 2: Theta***

- Allow parents that are non-neighbors in the grid to be used during search
- Standard A*
  - $g(y) = g(x) + c(x, y)$
  - Insert $y$ with estimate
    \[ f(y) = g(x) + c(x, y) + h(y) \]
- Theta*
  - If parent($x$) is visible from $y$, insert $y$ with estimate
    \[ f(y) = g(\text{parent}(x)) + c(\text{parent}(x), y) + h(y) \]
**Theta**\(^*\) *works*!

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**Theta**\(^*\) path, I think 😊

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Shortest path
Theta* works!

[Nash, AIGameDev 2010]
THE OPTIMAL PATH

• Polygonal path: sequence of connected straight lines
• Inner vertex of polygonal path: vertex that is not beginning or end
• Theorem: assuming (closed) polygonal obstacles, shortest path is a polygonal path whose inner vertices are vertices of obstacles
Proof of Theorem

- Suppose for contradiction that shortest path is not polygonal
- Obstacles are polygonal \( \Rightarrow \)
  - \( \exists \) point \( p \) in interior of free space such that “path through \( p \) is curved”
- \( \exists \) disc of free space around \( p \)
- Path through disc can be shortened by connecting points of entry and exit
**Proof of Theorem**

- Path is polygonal!
- Vertex cannot lie in interior of free space, otherwise we can do the same trick
- Vertex cannot lie on an edge, otherwise we can do the same trick □
How would we define a graph on which A* would be optimal?
**Visibility Graph**

Vertices = vertices of polygons and $s, t$
Edges = all $(x, y)$ such that $y$ is visible from $x$
VISIBILITY GRAPH

• **Poll 2:** Let $n$ be the total number of vertices of all polygons. How many edges will the optimal path in the visibility graph traverse in the worst case?

  1. $\Theta(\sqrt{n})$
  2. $\Theta(n)$
  3. $\Theta(n^2)$
  4. $\Theta(n^3)$
SUMMARY

• Terminology and algorithms:
  o Cell decomposition
  o Resolution completeness
  o Theta*

• Theorems:
  o A* is optimally efficient
  o Geometry of shortest path with polygonal obstacles

• Big ideas:
  o A* can be modified to work well in continuous spaces