Lec 11: Finite Element Discretization
15-769: Physically-based Animation of Solids and Fluids (F23)
Recap: Strong Form — Conservation of Mass

• Density: \[ R(X, t) = \lim_{\epsilon \to +0} \frac{\text{mass}(B^t_\epsilon)}{\text{volume}(B^t_\epsilon)} = \lim_{\epsilon \to +0} \frac{\text{mass}(B^t_\epsilon)}{\int_{B^t_\epsilon} dx} \]

\[ B^t_\epsilon \text{ is the ball of radius } \epsilon \text{ surrounding an arbitrary } X \in \Omega^0 \]

• Conservation of Mass: \( B^t_\epsilon \) is constant over time

\[ \text{mass}(B^t_\epsilon) = \int_{B^t_\epsilon} R(X(x), t) dx = \int_{B^0_\epsilon} R(X, t) J(X, t) dX = \text{mass}(B^0_\epsilon) = \int_{B^0_\epsilon} R(X, 0) dX \]

\[ \forall B^0_\epsilon \subset \Omega^0 \text{ and } t \geq 0 \]

\[ R(X, t) J(X, t) = R(X, 0), \quad \forall X \in \Omega^0 \text{ and } t \geq 0 \]

• Automatically satisfied in Lagrangian methods
• Needs to be explicitly considered in Eulerian methods
Recap: Strong Form — Conservation of Momentum

- Surface force defined via traction: \( \text{force}_S(B^0_\epsilon) = \int_{\partial B^0_\epsilon} T(X,N)ds(X) \)

- Stress (based on Cauchy’s Stress Theorem) (first Piola-Kirchoff stress) \( P(\cdot, t): \Omega^0 \rightarrow \mathbb{R}^{d \times d} \)

\[
T(X,N,t) = P(X,t)N.
\]

Applying Newton’s 2nd Law on \( B^0_\epsilon \):

\[
\int_{B^0_\epsilon} R(X,0) \frac{\partial V}{\partial t}(X,t)dX = \int_{\partial B^0_\epsilon} P(X,t)N(X)ds(X) + \int_{B^0_\epsilon} R(X,0)A^{ext}(X,t)dX, \quad \forall \ B^0_\epsilon \subset \Omega^0 \text{ and } t \geq 0
\]

Applying Divergence Theorem and extract the integrands:

\[
R(X,0) \frac{\partial V}{\partial t}(X,t) = \nabla^X \cdot P(X,t) + R(X,0)A^{ext}(X,t), \quad \forall \ X \in \Omega^0 \text{ and } t \geq 0.
\]
Recap: Weak Form Derivation

Ignoring external force for simplicity

For arbitrary test function \( Q(\cdot, t) : \Omega^0 \to \mathbb{R}^d \), compute the dot product to both sides and integrate

\[
\int_{\Omega^0} R(X, 0)Q(X, t) \cdot A(X, t) dX = \int_{\Omega^0} Q(X, t) \cdot (\nabla^X \cdot P(X, t)) dX, \quad \forall \ Q(\cdot, t) : \Omega^0 \to \mathbb{R}^d \text{ and } t \geq 0
\]

In index notation:

\[
\int_{\Omega^0} R(X, 0)Q_i(X, t)A_i(X, t) dX = \int_{\Omega^0} Q_i(X, t)P_{ij, j}(X, t) dX.
\]

Apply Integration By Parts and Divergence Theorem to RHS:

\[
\int_{\Omega^0} R(X, 0)Q_i(X, t)A_i(X, t) dX
\]

\[
= \int_{\partial \Omega^0} Q_i(X, t)P_{ij}(X, t)N_j(X) d\gamma(X) - \int_{\Omega^0} Q_{i, j}(X, t)P_{ij}(X, t) dX.
\]

— equivalent to strong form as it needs to hold for arbitrary \( Q \)
Recap: Weak Form Discretization

Looking at a specific moment $t = t^n$, Approximating $Q$ and $x$ by sampling and interpolation:

$$Q_i(X, t^n) \approx \sum_a Q_{a|i}(t^n) N_a(X) = \sum_a Q_{a|i}^n N_a(X),$$

$$x_i(X, t^n) \approx \sum_b x_{b|i}(t^n) N_b(X) = \sum_b x_{b|i}^n N_b(X).$$

Then

$$A_i(X, t^n) \approx \sum_b A_{b|i}(t^n) N_b(X) = \sum_b A_{b|i}^n N_b(X).$$

$$\int_{\Omega^0} R^0(X) Q_{i}^n(X) A_{i}^n(X) dX = \int_{\partial \Omega^0} Q_{i}^n(X) T^n_i(X) ds(X) - \int_{\Omega^0} Q_{i,j}^n(X) P_{ij}^n(X) dX$$

where

$$M_{ab} = \int_{\Omega^0} R(X, 0) N_a(X) N_b(X) dX$$

is the mass matrix.
Recap: Weak Form Discretization (Cont.)

\[
M_{ab}Q_{a|i}^n A_{b|i}^n = \int_{\partial \Omega^0} Q_{a|i}^n N_a(X) T_i(X, t^n) ds(X) - \int_{\Omega^0} Q_{a|i}^n N_{a,j}(X) P_{ij}(X, t^n) dX
\]

• Our discretization limit our solution space to the d*n interpolation functions.
• Test function can be chosen as

\[
Q_{a|i}^n = \begin{cases} 
1, & a = \hat{a} \text{ and } i = \hat{i} \\
0, & \text{otherwise}
\end{cases}
\]

to obtain \(nd\) equations

\[
M_{ab}A_{b|i}^n = \int_{\partial \Omega^0} N_{\hat{a}}(X) T_{\hat{i}}(X, t^n) ds(X) - \int_{\Omega^0} N_{\hat{a},j}(X) P_{ij}(X, t^n) dX.
\]

By applying mass lumping and zero traction boundary condition \(T(X, t) = 0\), we get

\[
M(x^{n+1} - (x^n + \Delta t v^n)) - \Delta t^2 f(x^{n+1}) = 0
\]

with elasticity force \(f(x^{n+1})\) obtained by evaluating \(- \int_{\Omega^0} N_{\hat{a},j}(X) P_{ij}(X, t) dX\)
Piecewise Linear Displacement Field

- We partition the space into simplex elements
- In 2D: Triangle meshes

\[ \Omega = \{ \} \]

Approximate the world-space coordinates (DOF) via interpolation:

\[ \hat{x}(X) = x(X_1)N_1(X) + x(X_2)N_2(X) + x(X_3)N_3(X) \]

Linear interpolation functions gives a linear Displacement Field:

\[ u = \hat{x}(X) - X \]
Parameter Space with Barycentric Coordinates

Let $\beta, \gamma \in [0, 1]$ and $\beta + \gamma = 1$, we can use them to express the material space coordinate of an arbitrary point $X$ in element $X_1X_2X_3$ as

$$X(\beta, \gamma) = X_1 + \beta(X_2 - X_1) + \gamma(X_3 - X_1) = (1 - \beta - \gamma)X_1 + \beta X_2 + \gamma X_3.$$ 

$- X$ is a linear function of $\beta$ and $\gamma$, so should $\hat{x}(\beta, \gamma)$

Similarly, in world space:

$$x(\beta, \gamma) \approx \hat{x}(\beta, \gamma) = x_1 + \beta(x_2 - x_1) + \gamma(x_3 - x_1) = (1 - \beta - \gamma)x_1 + \beta x_2 + \gamma x_3,$$

where we denote $x(X_i)$ as $x_i$. This indicates that

$$N_1(\beta, \gamma) = 1 - \beta - \gamma, \quad N_2(\beta, \gamma) = \beta, \quad N_3(\beta, \gamma) = \gamma.$$
Accurate Boundary Representation

- The discretized boundary converges to the continuous one with more nodes
- The partition clearly defines the boundary, no fuzziness like particle methods

\[ \Omega, \quad \{ \} \]

- Convenient for accurately handling boundary conditions and contact
- Partition of unity satisfied everywhere inside the domain

\[ N_1(\beta, \gamma) + N_2(\beta, \gamma) + N_3(\beta, \gamma) = 1 \quad \forall \beta, \gamma \in [0, 1] \text{ and } \beta + \gamma = 1 \]
*Surface Extraction for Particle Methods*

- Traditional method:
  - Construct signed distance field (SDF) by treating particles as spheres
  - SDF -> Surface mesh via marching cube (possibly with some smoothing)
- Wang et al. [2019], Qu et al. [2022]: use Voronoi/Power diagram
Mass Matrix (Discretization)

\[ M_{ab} = \int_{\Omega^0} R(\mathbf{X}, 0) N_a(\mathbf{X}) N_b(\mathbf{X}) d\mathbf{X} \]

With the solid domain discretized into triangles \( \mathcal{T} \), we have

\[ M_{ab} = \sum_{e \in \mathcal{T}} \int_{\Omega^0_e} R(\mathbf{X}, 0) N_a(\mathbf{X}) N_b(\mathbf{X}) d\mathbf{X}, \]

where \( \Omega^0_e \) represents the material space of triangle \( e \).

\( N_i \) is nonzero only on the incident triangles of node \( i \)

Let us change the integration variable from \( \mathbf{X} \) to \( (\beta, \gamma) \), which gives

\[ \int_{\Omega^0_e} R(\mathbf{X}, 0) N_a(\mathbf{X}) N_b(\mathbf{X}) d\mathbf{X} = \int_0^1 \int_0^{1-\beta} R(\beta, \gamma, 0) N_a(\beta, \gamma) N_b(\beta, \gamma) | \det(\frac{\partial \mathbf{X}}{\partial(\beta, \gamma)})| d\gamma d\beta. \]
Mass Matrix (Calculation)

\[ \int_{\Omega_e^0} R(X, 0)N_a(X)N_b(X)dX = \int_0^1 \int_0^{1-\beta} R(\beta, \gamma, 0)N_a(\beta, \gamma)N_b(\beta, \gamma) \left| \det\left( \frac{\partial X}{\partial (\beta, \gamma)} \right) \right| d\gamma d\beta. \]

For simplicity, let us denote the vertices of this triangle \( e \) as \( X_1, X_2, \) and \( X_3, \) and then we have

\[ \left| \det\left( \frac{\partial X}{\partial (\beta, \gamma)} \right) \right| = \left| \det([X_2 - X_1, X_3 - X_1]) \right| = 2A_e, \]

where \( A_e \) is the area of triangle \( e. \) Here, \( N_a \) and \( N_b \) takes \( 1 - \beta - \gamma, \beta, \) or \( \gamma \) depending on the vertex indices \( a \) and \( b. \) For example, if \( a \) and \( b \) correspond to the 2nd and 3rd vertices of triangle \( e, \) then \( N_a = \beta \) and \( N_b = \gamma. \)
Mass Matrix (Calculation, Cont.)

\[
\int_{\Omega_e^0} R(X,0)N_a(X)N_b(X)dX = \int_0^1 \int_0^{1-\beta} R(\beta, \gamma, 0)N_a(\beta, \gamma)N_b(\beta, \gamma) \left| \text{det} \left( \frac{\partial X}{\partial (\beta, \gamma)} \right) \right| d\gamma d\beta.
\]

\[
\int_{\Omega_e^0} R(X,0)N_a(X)N_b(X)dX = 2RA_e \int_0^1 \int_0^{1-\beta} \beta \gamma d\gamma d\beta
\]

\[
= 2RA_e \int_0^1 \frac{1}{2} \beta \gamma^2 \bigg|_{\gamma=0}^{\gamma=1-\beta} d\beta
\]

\[
= RA_e \int_0^1 \beta(1-\beta)^2 d\beta
\]

\[
= RA_e \left( \frac{\beta^2}{2} - \frac{2\beta^3}{3} + \frac{\beta^4}{4} \right) \bigg|_{\beta=0}^{\beta=1} = \frac{1}{12} RA_e
\]

(Assuming uniform density R)
Lumped Mass Matrix (Discretization)

With mass lumping, $M_{ab}^{\text{lump}} = \delta_{ab} \sum_c M_{ac}$, which means

$$M_{aa}^{\text{lump}} = \sum_{e \in T} \sum_{b \in \mathcal{V}} \int_{\Omega_e} R(\mathbf{X}, 0) N_a(\mathbf{X}) N_b(\mathbf{X}) d\mathbf{X},$$

where $\mathcal{V}$ contains all the nodes of the mesh, and all off-diagonal entries of $M^{\text{lump}}$ are 0. Similarly, due to the locality of $N$, for each triangle element, $b$ only needs to traverse all the 3 triangle vertices, i.e.
Lumped Mass Matrix (Calculation)

\[
M_{aa}^{\text{lump}} = \sum_{e \in \mathcal{T}(a)} 2RA_e \left( \int_0^1 \int_0^{1-\beta} \beta(1 - \beta - \gamma) d\gamma d\beta + \int_0^1 \int_0^{1-\beta} \beta^2 d\gamma d\beta \right) + \int_0^1 \int_0^{1-\beta} \beta \gamma d\gamma d\beta
\]

\[
= \sum_{e \in \mathcal{T}(a)} 2RA_e \int_0^1 \int_0^{1-\beta} \beta d\gamma d\beta = \sum_{e \in \mathcal{T}(a)} 2RA_e \int_0^1 \beta \gamma|_{\gamma=0}^{\gamma=1-\beta} d\beta
\]

\[
= \sum_{e \in \mathcal{T}(a)} 2RA_e \int_0^1 \beta(1 - \beta) d\beta = \sum_{e \in \mathcal{T}(a)} 2RA_e \left( \frac{\beta^2}{2} - \frac{\beta^3}{3} \right)|_{\beta=0}^{\beta=1}
\]

\[
= \sum_{e \in \mathcal{T}(a)} \frac{1}{3} RA_e,
\]

where \( \mathcal{T}(a) \) denotes the set of triangles incident to node \( a \).
Elasticity

Strain and Stress Calculation

\[ \int_{\Omega} N_{\hat{a},j}(X) P_{i\hat{j}}(X, t^n) dX = \int_{\Omega} (P(X, t^n) \nabla^X N_{\hat{a}}(X))_i dX \]  
(Vector notation)

\[ = \sum_{e \in T} \int_{\Omega^0_e} (P(X, t^n) \nabla^X N_{\hat{a}}(X))_i dX. \]  
(Triangulation)

Analogously, this summation also only needs to involve the incident triangles of node \( \hat{a} \).

\[ \mathbf{P} = \frac{\partial \Psi}{\partial \mathbf{F}} \]  
can be calculated with \( \mathbf{F} \):

\[ \mathbf{F} = \frac{\partial \mathbf{x}}{\partial (\beta, \gamma)} \left( \frac{\partial \mathbf{X}}{\partial (\beta, \gamma)} \right)^{-1} \approx \frac{\partial \mathbf{x}}{\partial (\beta, \gamma)} \left( \frac{\partial \mathbf{X}}{\partial (\beta, \gamma)} \right)^{-1} \]

\[ = [\mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_1][\mathbf{X}_2 - \mathbf{X}_1, \mathbf{X}_3 - \mathbf{X}_1]^{-1}, \]

piecewise constant in \( \Omega^0 \), so does \( \mathbf{P} \)
Elasticity

Kernel Gradient

\[ \sum_{e \in T} \int_{\Omega^e} (P(X, t^n) \nabla^X N_{\hat{a}}(X))_i \, dX \]

\[ \nabla^X N_1(X) = \frac{\partial (1 - \beta - \gamma)}{\partial X} = \left( \frac{\partial (1 - \beta - \gamma)}{\partial (\beta, \gamma)} \left( \frac{\partial X}{\partial (\beta, \gamma)} \right)^{-1} \right)^T \]
\[ = \left( [1, 0][X_2 - X_1, X_3 - X_1]^{-1} \right)^T \]

\[ \nabla^X N_2(X) = \frac{\partial \gamma}{\partial X} = \left( \frac{\partial \gamma}{\partial (\beta, \gamma)} \left( \frac{\partial X}{\partial (\beta, \gamma)} \right)^{-1} \right)^T = ([1, 0][X_2 - X_1, X_3 - X_1]^{-1})^T \]

\[ \nabla^X N_3(X) = \frac{\partial \gamma}{\partial X} = \left( \frac{\partial \gamma}{\partial (\beta, \gamma)} \left( \frac{\partial X}{\partial (\beta, \gamma)} \right)^{-1} \right)^T = ([0, 1][X_2 - X_1, X_3 - X_1]^{-1})^T \]

\[ \frac{\partial \Psi_e}{\partial x_{\hat{a}}} = \frac{\partial \Psi_e}{\partial F} \frac{\partial F}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x_{\hat{a}}} = P \frac{\partial \hat{x}}{\partial \hat{x}} N_{\hat{a}} = P \nabla^X N_{\hat{a}} \]

\[ \frac{\partial [F_{11}, F_{21}, F_{12}, F_{22}]^T}{\partial [x_1^T, x_2^T, x_3^T]^T} = \begin{bmatrix} B_{11} & B_{21} \\ -B_{11} - B_{21} & B_{11} & B_{21} \\ -B_{12} - B_{22} & B_{12} & B_{22} \end{bmatrix} \in \mathbb{R}^{4 \times 6} \]

\[ B = [X_2 - X_1, X_3 - X_1]^{-1} \]
Elasticity

Remarks

\[
\int_{\Omega_0} N_{\hat{a},j}(\mathbf{X}) P_{ij}(\mathbf{X}, t^n) d\mathbf{X} = \sum_{e \in \mathcal{T}} \int_{\Omega_0^e} (\mathbf{P}(\mathbf{X}, t^n) \nabla^X N_{\hat{a}}(\mathbf{X}))_{i}^e d\mathbf{X}
\]

\[
= \sum_{e \in \mathcal{T}} \int_{\Omega_0^e} \left( \frac{\partial \Psi_e}{\partial x_{\hat{a}}} \right)_i d\mathbf{X}
\]

\[
= \sum_{e \in \mathcal{T}} A_e \left( \frac{\partial \Psi_e}{\partial x_{\hat{a}}} \right)_i,
\]

- Energy:

\[
\sum_{e \in \mathcal{T}} A_e \Psi_e, \text{ which is } \int_{\Omega_0} \Psi(\mathbf{X}) d\mathbf{X} \text{ before spatial discretization.}
\]

Remark 19.2. Linear FEM refers to \( x \) is a piecewise linear function of \( \mathbf{X} \), but the elasticity model can still be nonlinear, i.e. \( \mathbf{P} \) can be a nonlinear function of \( \mathbf{F} \).
Higher-Order FEM

\textit{e.g. \cite{Ferguson2023}}

\begin{align*}
  P_1 \text{ coarse} & \quad \text{(2m 47s)} \\
  P_1 \text{ time budgeted} & \quad \text{(6h 7m 12s)} \\
  P_2 & \quad \text{(6h 19m 52s)} \\
  P_1 & \quad \text{(2d 14h 13m 0s)}
\end{align*}
FEM with Non-Simplex Elements
e.g. Chen et al. [2023]


Boundary Conditions

\[ R(\mathbf{X}, 0) \frac{\partial \mathbf{V}}{\partial t}(\mathbf{X}, t) = \nabla^X \cdot \mathbf{P}(\mathbf{X}, t) + R(\mathbf{X}, 0) \mathbf{A}^{\text{ext}}(\mathbf{X}, t), \]

\[ \forall \mathbf{X} \in \Omega^0 \text{ and } t \geq 0; \]

\[ \mathbf{x} = \mathbf{x}_D(\mathbf{X}, t), \]

\[ \forall \mathbf{X} \in \Gamma_D \text{ and } t \geq 0; \]

\[ \mathbf{P}(\mathbf{X}, t)\mathbf{N}(\mathbf{X}) = \mathbf{T}_N(\mathbf{X}, t), \]

\[ \forall \mathbf{X} \in \Gamma_N \text{ and } t \geq 0. \]

Here \( \Gamma_N \) and \( \Gamma_D \) are the Neumann and Dirichlet boundaries respectively, \( \Gamma_N \cup \Gamma_D = \partial \Omega_0, \Gamma_N \cap \Gamma_D = \emptyset \), and \( \mathbf{x}_D \) and \( \mathbf{T}_N \) are given. After we derive the weak form of the momentum conservation (Equation 18.1 1st line), the boundary term \( \int_{\partial \Omega_0} Q_i(\mathbf{X}, t) T_i(\mathbf{X}, t) ds(\mathbf{X}) \) can be separately considered for Dirichlet and Neumann boundaries:

\[ \int_{\partial \Omega_0} Q_i(\mathbf{X}, t) T_i(\mathbf{X}, t) ds(\mathbf{X}) \]

\[ = \int_{\Gamma_D} Q_i(\mathbf{X}, t) T_{D|i}(\mathbf{X}, t) ds(\mathbf{X}) + \int_{\Gamma_N} Q_i(\mathbf{X}, t) T_{N|i}(\mathbf{X}, t) ds(\mathbf{X}). \]
Dirichlet Boundary Conditions (Discretized)

Due to the accurate boundary resolution of FEM, to enforce Dirichlet boundary conditions, we just need to constrain the world-space coordinates of the boundary nodes to the prescribed values:

\[ \hat{x}(X_i) = x_D(X_i) \quad \forall \; X_i \in \Gamma_D. \]

- Ignore the integral

\[
\int_{\partial \Omega^0} Q_i(X, t)T_i(X, t)ds(X) = \int_{\Gamma_D} Q_i(X, t)T_{D|i}(X, t)ds(X) + \int_{\Gamma_N} Q_i(X, t)T_{N|i}(X, t)ds(X).
\]

- Can apply to inner nodes although may not be physical
Neumann Boundary Conditions (Discretized)

\[
\int_{\Gamma_N} N_{\hat{a}}(X) T_i(X, t^n) ds(X) = \sum_{e \in \mathcal{T}} \int_{\partial \Omega_e \cap \Gamma_N} N_{\hat{a}}(X) T_i(X, t^n) ds(X)
\]

For any boundary node \( \hat{a} \), in 2D there will be at most 2 incident triangles to be considered in the integration for linear shape functions. For the case with 2 incident triangles, let us look at one of the integral. Without loss of generality, we can assume \( N_{\hat{a}} = \beta \) (\( X_{\hat{a}} \) corresponds to \( X_2 \) in triangle \( e \)), and that \( X_3 \) is the other node of \( e \) on the boundary edge. Then, switching the integration variables to \( \beta \) gives us

\[
\int_{\partial \Omega_e \cap \Gamma_N} N_{\hat{a}}(X) T_i(X, t^n) ds(X) = \int_0^1 \beta T_i(\beta X_2 + (1 - \beta) X_3, t^n) \frac{\partial s}{\partial \beta} |d\beta|.
\]

Here \( |\frac{\partial s}{\partial \beta}| \) is simply the edge length \( ||X_2 - X_3|| \). If \( T \) is constant over the boundary at \( t^n \), we can compute

\[
T_i^n \int_0^1 \beta \frac{\partial s}{\partial \beta} |d\beta| = \frac{1}{2} ||X_2 - X_3|| T_i^n.
\]
Neumann Boundary Conditions (Discretized)

Remark Here we see that the specified traction in standard Neumann boundary conditions is independent of \(x\), which makes deriving the potential energy trivial even in the continuous setting for varying Neumann forces over the domain:

\[
\int_{\Gamma_N} \mathbf{x}(\mathbf{X}) \cdot \mathbf{T}(\mathbf{X}, t^n) ds(\mathbf{X}).
\]

To verify it, we can replace \(\mathbf{x}(\mathbf{X})\) with \(\dot{\mathbf{x}}(\mathbf{X}) = N_{\dot{\alpha}}(\mathbf{X}) \mathbf{x}_{\dot{\alpha}} + ...\) for spatial discretization, and then take its derivative w.r.t. \(x_{\dot{\alpha}}\) to obtain the force integral term in the discrete weak form:

\[
\frac{\partial}{\partial x_{\dot{\alpha}}} \int_{\Gamma_N} \dot{\mathbf{x}}(\mathbf{X}) \cdot \mathbf{T}(\mathbf{X}, t^n) ds(\mathbf{X}) = \int_{\Gamma_N} N_{\dot{\alpha}}(\mathbf{X}) \mathbf{T}(\mathbf{X}, t^n) ds(\mathbf{X}).
\]
Next Lecture: Self-Contact

\[
\int_{\partial\Omega^0} Q_i(X, t)T_i(X, t)ds(X)
\]

\[
= \int_{\Gamma_D} Q_i(X, t)T_D|_i(X, t)ds(X) + \int_{\Gamma_N} Q_i(X, t)T_N|_i(X, t)ds(X)
\]

\[
+ \int_{\Gamma_C} Q_i(X, t)T_C|_i(X, t)ds(X),
\]
Image Sources

- https://dl.acm.org/doi/10.1145/3340259