PROBLEM SET 7

Due: 5 pm, Fri Nov. 18, email the pdf to toolkit2016homework@gmail.com

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Homework policy: Exactly the same as last time.

Solve 3 of the first 4 problems, and then problems 5 and 6.

1. It is well known that the 2SAT problem, which asks whether an input 2CNF formula with clauses of width *at most* 2 (unary clauses are allowed), is polynomial time solvable. However, the optimization version, Max 2SAT, where the goal is to maximize the number of satisfied clauses, is NP-hard. (Indeed, Max 2SAT is a more general problem than Max Cut.)

   (a) Show that, similarly to Max Cut, the Max 2SAT problem can be expressed as an *integer quadratic program* in which the only constraints are $y_i \in \{1, -1\}$ and the objective function is a quadratic in the y_i.
   (Hint: It may help to introduce a variable y_0 to designate which of 1 or -1 indicates "true".)

   (b) Derive a factor $\alpha_{\mathrm{GW}} \approx 0.87856$ approximation algorithm for Max 2SAT.

   (c) Consider the restricted class of Max 2SAT where each clause has exactly two distinct literals, and where each variable appears negated and unnegated in the same number of clauses. For such instances, give a ξ-approximation, where

   $$\xi = \min_{0 \le \theta \le \pi} \frac{2 + \frac{2\theta}{\pi}}{3 - \cos\theta} \approx 0.94394\,.$$

2. Suppose we have n jobs $J_1, J_2, \ldots, J_n$ which are to be scheduled in time slots $1, 2, \ldots, n$. If we only have constraints of the $J_r < J_s$ (which means J_r must be scheduled earlier than J_s), then it is easy to tell if a schedule meeting all the constraints exists. Now suppose we have several constraints of the form "J_s must be scheduled between J_r and J_t which may be in either order (i.e., either $J_r < J_s < J_t$ or $J_t < J_s < J_r$)." Now ascertaining the existence of a schedule obeying all these constraints becomes NP-hard. This exercise will investigate the approximability of this problem.

   (a) Suppose we pick a random schedule of the jobs. What fraction of constraints do we satisfy in expectation? What approximation ratio does this imply?

   For the rest of the exercise, assume that we are promised to be given a satisfiable instance where a schedule obeying all the constraints exists. The goal is to efficiently find a schedule satisfying a good fraction of the constraints.

   (b) Prove that the following quadratic program is feasible:

   $$\begin{aligned} (y_i - y_j)^2 &\ge 1 \quad \text{for all } i, j,\ i \ne j \\ (y_r - y_s) \cdot (y_t - y_s) &\le 0 \quad \text{whenever } J_s \text{ between } J_r, J_t \text{ is a constrain} \end{aligned}$$

   (c) Write an SDP relaxation for the above quadratic program in the vector form.

(d) Consider the following rounding strategy: pick a random vector $r = (r_1, r_2, \ldots, r_n)$ where the r_i's are independent standard normal variables, and output the schedule based on the sorted order of $\langle v_i, r \rangle$.

Prove that this strategy satisfies in expectation at least $1/2$ the constraints.

3. Inspired by our success with Max Cut and Max 2SAT, it is natural to consider a generalization and approximate the quadratic program below:

$$\begin{aligned} \text{maximize} \quad & \sum_{1 \le i,j \le n} a_{ij} y_i y_j \\ \text{subject to} \quad & y_i \in \{1, -1\},\ i = 1, 2, \ldots, n. \end{aligned} \tag{1}$$

Before we speak of multiplicative approximation algorithms for the above, we need to be slightly careful because it is possible that the optimum is negative! (Can you think of a simple example where this is the case?)

To get around this predicament, let us restrict the coefficients in the objective function to satisfy the property that the matrix $A = (a_{ij})$ is symmetric and positive semidefinite.

(a) Write an SDP relaxation for the above quadratic program in the vector form.

(b) Compute an exact expression (in terms of inner products of the SDP vectors) for expected value of the objective function (1) for the solution found by random hyperplane rounding.

(c) Prove that if $M = (m_{ij})$ and $N = (n_{ij})$ are $n \times n$ psd matrices, then $M \circ N = (m_{ij} n_{ij})$ is also psd.

(d) If $Y = (y_{ij})$ is psd with $|y_{ij}| \le 1\ \forall i, j$, then the matrix $Z = (z_{ij})$ defined by $z_{ij} = \arcsin(y_{ij}) - y_{ij}$ is also psd.

Hint: The Taylor series expansion for $\arcsin(x)$ around 0 is

$$\arcsin(x) = x + \frac{1}{2 \cdot 3} x^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} x^5 + \cdots + \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n(2n+1)} x^{2n+1} + \cdots$$

(e) Prove that your expression in (b) is at least $\frac{2}{\pi}$ times the SDP optimum, thereby giving a factor $\frac{2}{\pi}$-approximation algorithm for this general problem.

4. In this exercise, we will revisit the rounding algorithm for coloring 3-colorable graphs to obtain an improved bound.

Recall we have an n-vertex 3-colorable graph $G = (\{1, 2, \ldots, n\}, E)$ with degree bound Δ. Assume we have solved the SDP from lecture and found unit vectors $\{v_i\}$ such that $\langle v_i, v_j \rangle = -1/2$ whenever $(i, j) \in E$.

(a) Suppose we pick a random vector $r = (r_1, r_2, \ldots, r_n)$ where the r_i's are independent standard normal variables, and define $S = \{i \mid \langle v_i, r \rangle \ge \tau\}$ for some τ. Prove that

$$\mathbb{E}[|S|] \ge n\phi(\tau) \left(\frac{1}{\tau} - \frac{1}{\tau^3} \right)$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is the density function of the standard normal variable $N(0, 1)$.

Hint: You may use Proposition 4.3 of the Lecture 2 notes from 2013.

(b) Let T denote the number of edges in the subgraph of G induced by S. Prove that

$$\mathbb{E}[T] \le n\Delta \frac{\phi(2\tau)}{4\tau}\,.$$

(c) Put $\tau := \sqrt{\frac{2\ln\Delta}{3}}$. Using the above, show that one can efficiently find an independent set in G of expected size at least $\Omega\left(\frac{n}{\Delta^{1/3}\sqrt{\ln\Delta}}\right)$.

(d) Show that one can efficiently find a proper coloring of G that uses $O(\Delta^{1/3}\sqrt{\ln\Delta}\ln n)$ colors.

(e) Finally, argue that one can efficiently find a proper coloring of a 3-colorable graph using $O(n^{1/4}(\ln n)^{9/8})$ colors.

5. Recall that we gave two formulations of the Lovász theta function in class. Let use denote by $\theta_L(G)$ Lovász's original formulation for a graph $G = (\{1, 2, \ldots, n\}, E)$:

$$\theta_L(G) = \min_{\{v_i\}_{1\le i\le n}} \max_{1\le i\le n} \frac{1}{v_{i1}^2} \tag{2}$$

where the outer minimum is over the choice of unit vectors v_i obeying $\langle v_i, v_j\rangle = 0$ whenever $(i,j) \notin E$. We also have the (strict) vector chromatic number formulation $\theta_{\text{svc}}(G)$ as the optimum of the following semidefinite program:

$$\begin{aligned} \text{minimize} \quad & k \\ \langle v_i, v_i\rangle = 1 \quad & i = 1, 2, \ldots, n \\ \langle v_i, v_j\rangle = \frac{-1}{k-1} \quad & (i,j) \notin E \end{aligned}$$

(a) Prove that $\alpha(G) \le \theta_L(G) \le \overline{\chi}(G)$ where $\alpha(G)$ is the size of the largest independent set in G, and $\overline{\chi}(G)$ is the minimum number of cliques required to cover every vertex of G.

(b) Formalize the "umbrella" argument alluded to in lecture to establish $\theta_L(G) \le \theta_{\text{svc}}(G)$.

(c) Define a variant $\theta'_L(G)$ of $\theta_L(G)$ by restricting the outer minimum in (2) over all unit vectors v_i obeying $\langle v_i, v_j\rangle = 0$ whenever $(i,j) \notin E$ and $v_{11} = v_{21} = \cdots = v_{n1}$ (i.e., they all have equal first coordinate).

Prove that $\theta_{\text{svc}}(G) = \theta'_L(G)$. (You probably proved one direction already in part (b).)

(d) Prove that there is a symmetric matrix $A = (a_{ij})$ with $a_{ij} = 1$ for $(i,j) \notin E$ and $a_{ii} = 1$ for $i = 1, 2, \ldots, n$ with largest eigenvalue $\lambda_{\max}(A) \le \theta_L(G)$.

<u>Hint</u>: Take vectors v_i attaining the minimum in (2) and define $a_{ij} = 1 - \frac{\langle v_i, v_j\rangle}{v_{i1}v_{j1}}$ for $i \ne j$.

(e) For any symmetric matrix $A = (a_{ij})$ with $a_{ij} = 1$ for $(i,j) \notin E$ and $a_{ii} = 1$ for $i = 1, 2, \ldots, n$, prove that $\theta'_L(G) \le \lambda_{\max}(A)$.

<u>Hint</u>: $\lambda_{\max} I - A \gtrsim 0$ and use its Cholesky decomposition to define the unit vectors v_i with common first coordinate and $\langle v_i, v_j\rangle = 0$ for $(i,j) \notin E$.

(f) Deduce $\theta_L(G) = \theta_{\text{svc}}(G)$ based on the above.

6. We would like to know the best possible rate of a binary code (not necessarily linear) of minimum distance at least d. Equivalently, we would like to know the number $A(n,d)$, defined to be the largest possible cardinality of a set $C \subseteq \mathbb{F}_2^n$ such that every distinct pair

$v, w \in C$ has $\triangle(v, w) \geq d$. It's a famous open problem to analyze the asymptotics of $A(n, d)$; this problem illustrates the best known approach. We remark that the problem is actually an explicit instance of the "Max-Independent-Set" problem (consider the graph with vertex set $\mathbb{F}_2^n$ and an edge joining any v, w with $\triangle(v, w) \leq d - 1$).

(a) One SDP relaxation for this problem is the following

$$
\begin{aligned}
\text{maximize } & \sum_{v,w \in \mathbb{F}_2^n} y_{vw} \\
\text{subject to: } & \text{the } 2^n \times 2^n \text{ matrix } Y = (y_{vw}) \text{ is symmetric and positive semidefinite} \\
& \mathrm{Tr}(Y) := \sum_{v \in \mathbb{F}_2^n} y_{vv} = 1 \\
& y_{vw} \geq 0 \text{ for all } v, w \in \mathbb{F}_2^n \\
& y_{vw} = 0 \text{ for all } v, w \in \mathbb{F}_2^n \text{ with } 0 < \triangle(v, w) < d.
\end{aligned}
$$

Show this is indeed a relaxation; i.e., the optimal value of this SDP is at least $A(n, d)$.

(Remark: In fact, this relaxation is another one of the many equivalent avatars of the Lovász theta function (for the graph $G = (\mathbb{F}_2^n, \{(v, w) \mid \triangle(v, w) < d\})$ whose independence number equals $A(n, d)$), though you don't need to show this, and you shouldn't appeal to this fact either.)

(b) It can be shown (and if we get around to studying Fourier analysis, perhaps we will show), that the above SDP relaxation is *equivalent* to the following much simpler LP relaxation:

$$
\begin{aligned}
\text{maximize } & \sum_{k=0}^{n} \sigma_k \\
\text{subject to: } & \sigma_0 = 1 \\
& \sigma_1, \sigma_2, \ldots, \sigma_{d-1} = 0 \\
& \sigma_d, \sigma_{d+1}, \ldots, \sigma_n \geq 0 \\
& \sum_{k=0}^{n} K_j(k) \sigma_k \geq 0 \quad \text{for all } 0 \leq j \leq n.
\end{aligned}
$$

Here the number $K_j(k)$ is defined by

$$
K_j(k) = \sum_{\substack{y \in \mathbb{F}_2^n \\ |y| = j}} (-1)^{x \cdot y},
$$

where $x \in \mathbb{F}_2^n$ is any string satisfying $|x| = k$ (doesn't matter which).

Prove that this LP is indeed a relaxation; i.e., its optimal value is at least $A(n, d)$.

(Hint: consider the average number of codewords at distance k, or some such thing.)