Problem Set 1

Due: Noon, Monday September 19, email the pdf to toolkit2016homework@gmail.com

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Homework policy: LaTeX typesetting with pdf output is mandatory; poor LaTeX form on any solution will lead to a loss of points. Please try to do the homework by yourself. If you get stuck, working in a group of two is okay; maybe three, max. You may not discuss the homework with anyone outside of the class. You may use computational mathematics programs to assist you. You may not search the Internet for answers to the *specific* questions in the homework. On the other hand, you are welcome to search the Internet for general math concepts, and you are very strongly encouraged to search the Internet for any and all issues concerning LaTeX. Your homework pdf must end with an "Acknowledgments" section (like in a research paper) in which you acknowledge any assistance you received. Questions about the homework or other course material can be asked on Piazza. Finally, please note that the TAs will only be grading a subset of the problems.

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1. In lecture we stated the importance of the inequality $e^x \geq 1+x$ (for all $x \in \mathbb{R}$), which is useful for converting between products and sums when doing asymptotic estimation. It can also be helpful to have an upper bound on e^x for small x. Explicitly prove that $e^x \leq 1 + x + O(x^2)$ as $x \to 0$. For full points, do this by showing that $e^x \leq 1 + x + x^2$ for all $-1 \leq x \leq 1$.

   You may use the definition $e^t = \sum_{i=0}^{\infty} t^i/i!$ for all $t \in \mathbb{R}$.

2. (a) Give an extremely simple proof that $e^k > \frac{k^k}{k!}$ for all $k \in \mathbb{N}^+$; you should again use the definition of e^t above.

   (b) Give a short proof that $(\frac{n}{k})^k \leq \binom{n}{k} < (\frac{en}{k})^k$ for integers $0 < k < n$, a popular quick-and-dirty approximation.

3. (**Note: The quality of your LaTeX will be particularly judged in this problem.**)

   In this problem, your solutions should begin with a displayed equation of the following form:

   $$[\text{expression to be analyzed}] = \Theta(f(x)),$$

   where $f(x)$ is a "simple" function, and x is the asymptotic parameter (either "$n \to \infty$" or "$\varepsilon \to 0^+$"). You should then give an explanation of how you derived your answer (but a completely formal proof is not required). We give a sample problem/solution.

   (Sample problem.) Asymptotically simplify

   $$\sqrt{1+\varepsilon} - 1.$$

   (Sample solution.)

   $$\sqrt{1+\varepsilon} - 1 = \Theta(\varepsilon).$$

   The upper bound is immediate from $\sqrt{1+\varepsilon} \leq 1 + \varepsilon$. The lower bound follows from the fact that $\sqrt{1+\varepsilon} \geq 1 + \varepsilon/3$ for $0 \leq \varepsilon \leq 3$, which can be proved as follows: $(1+\varepsilon/3)^2 = 1 + \frac{2}{3}\varepsilon + \frac{1}{9}\varepsilon^2 \leq 1 + \varepsilon$, where we used $\varepsilon \leq 3 \implies \frac{1}{9}\varepsilon^2 \leq \frac{1}{3}\varepsilon$.

(a) Asymptotically simplify
$$\sqrt{n+1} - \sqrt{n}.$$

(b) Asymptotically simplify
$$\sum_{i=1}^{n} i^3.$$

(c) Asymptotically simplify
$$\frac{1}{1+\varepsilon} - (1-\varepsilon).$$

(d) Noting carefully the range of summation, asymptotically simplify
$$\sum_{i=n+1}^{n^2} \frac{1}{i}.$$

(e) Asymptotically simplify
$$\log_2\left(\frac{1}{\frac{1}{2}-\varepsilon}\right).$$

For this problem, your solution should be a degree-1 polynomial in ϵ with an error of $\pm\Theta(\varepsilon^2)$. You should treat $\varepsilon \to 0$ with ε potentially positive or negative.

(f) Assume $m \geq 1$ is the solution of $m^m = n$. Asymptotically express m as a function of n.

4. Recall the binary entropy function, defined for $0 \leq p \leq 1$ by $H_2(p) = p\log_2(\frac{1}{p}) + q\log_2(\frac{1}{q})$, where q denotes $1-p$. It is easy to see that $H_2(p)$ is symmetric about $p = \frac{1}{2}$, so it is sufficient to study $H_2(p)$ for $0 \leq p \leq \frac{1}{2}$.

   (a) Produce a plot of $H_2(p)$ for $0 \leq p \leq 1$ using your favorite computational mathematics tool. Insert it as a graphic into your homework pdf. State what tool you used.

   (b) Prove that $p\log_2(\frac{1}{p}) \leq H_2(p) \leq p\log_2(\frac{e}{p})$ for all $0 \leq p \leq \frac{1}{2}$. Then show how to deduce $H_2(p) \sim p\log_2(\frac{1}{p})$ as $p \to 0^+$.

   (c) Prove that $H_2(\frac{1}{2}-\varepsilon) = 1 - \Theta(\varepsilon^2)$ as $\varepsilon \to 0^+$.

5. Let $0 \leq k \leq n$ be integers and write $p = k/n$. In the lecture, Stirling's formula was used to show that
$$\binom{n}{k} = \Theta\left(\frac{1}{\sqrt{n}} \cdot 2^{H_2(p)n}\right) = \widetilde{\Theta}\left(2^{H_2(p)n}\right),$$
where H_2 is the binary entropy function from the previous question (assuming p and $1-p$ are "constant"). In this problem, you'll use a more lowbrow technique to show that $2^{H_2(p)n}$ is an upper bound (assuming $p \leq \frac{1}{2}$). In fact, this is even an upper bound on
$$V(n,k) := \sum_{i=0}^{k} \binom{n}{i}.$$

The quantity $V(n,k)$ arises frequently in theoretical computer science: it is the "volume of the radius-k Hamming ball in the n-dimensional hypercube"; i.e., it is the number of length-n binary strings with at most k 1's.

From now on, assume $k \leq n/2$.

(a) Show that $\binom{n}{0} \le \binom{n}{1} \le \binom{n}{2} \le \cdots \le \binom{n}{k}$. Hint: convert the ratio $\binom{n}{i}/\binom{n}{i-1}$ to factorials. (Note that we won't actually need this fact to bound $V(n,k)$. However, it does show that $\binom{n}{k} \le V(n,k) \le (k+1)\binom{n}{k}$, which should help orient you: $V(n,k)$ is $\widetilde{\Theta}(\cdot)$ of the largest term in the sum.)

(b) The sum $\sum_{i=0}^{k} \binom{n}{i}$ looks similar to the one arising in the Binomial Theorem, so it makes sense to try to relate them. Use this idea to show that

$$\theta^{pn} \cdot V(n,k) \le (1+\theta)^n$$

for any $0 < \theta \le 1$.

(c) Use calculus to find the most effective choice of θ, and deduce that $V(n,k) \le 2^{H_2(p)n}$.

6. Last week, one of the instructors faced the following problem in the course of his research. There is a sequence of reals $1 = a_1 \le a_2 \le a_3 \le \cdots \le a_n$. It is known to satisfy $a_{i+1} - a_i \le \sqrt{a_i}$ for all $1 \le i < n$. Prove that

$$\sum_{i=1}^{n-1} \frac{a_{i+1} - a_i}{a_i} \le O(\log n).$$

7. Include a bibliography at the end of your homework solutions containing 5 research papers that you are interested in reading. They should be drawn from your lifetime .bib file; use the `\nocite` command in LaTeX to get them into your references section without actually citing them.