This HW is going out a couple days before classes start and has a short deadline (end of week #1 of classes). This is so we can gauge your background, and also you can get a feel for the course and prepare accordingly. These problems should be solvable using ideas from your basic undergraduate algorithms, probability, and linear algebra courses.

Please solve the (non-exercise) problems without collaboration. You may discuss exercises with others. In general, you should solve the homework problems using only to material we refer you to (or put on the course page), and not books or other online resources. If you have reason to use any resources we point you to (except the course notes), you should cite them.

Homeworks will be due at 11:59pm on the due date on gradescope. Corrections and changes will appear on the course webpage and on Piazza, please check them regularly.

Exercises

Exercises are for fun and edification, please do not submit. But please do try to solve them, we may need ideas from there later in the course!

1. (Random Questions)
   (a) Given random variables (r.v.s) $X, Y$, show that $E[X + Y] = E[X] + E[Y]$ and $E[cX] = cE[X]$ and $\text{Var}(cX) = c^2 \text{Var}(X)$ for any constant $c$. If they are independent, then show that $E[XY] = E[X]E[Y]$ and $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$. Hence show that for independent $X_1, \ldots, X_n$, if each $X_i$ has mean $\mu$ and variance $\sigma^2$, then $\frac{\sum_{i=1}^{n} X_i}{n}$ has mean $\mu$ and variance $\sigma^2/n$.
   (b) At the latest CMU-Pitt game, $n$ students celebrate the CMU victory by throwing their caps up in the air. These caps fall back so that each person gets back one cap, but they are in uniformly random order (all $n!$ final permutations of the caps are equally likely). Show that the expected number of people who receive their own cap back is 1.
   (c) An airplane in Politesville has $n$ seats, and $n$ passengers assigned to these seats. The first passenger to board is a bit confused, and sits down at a uniformly random seat. The rest of the passengers do the following: when they board, if their assigned seat is free they sit in it, else (being too polite) they choose a uniformly random empty seat and sit in it. (a) What is the probability that the last person to board sits in their assigned seat? (b) What is the expected number of people who board to find their assigned seat occupied?
   (d) Suppose you have a coin with bias (i.e., probability of “heads”) being $p$. You repeatedly flip it until you get a heads—at that point you stop. What is the expected number of times until you stop? How does your answer change if you stop exactly when you have seen $k$ heads?

2. (Linear Algebra and Matrices) Given an $m \times n$ matrix $A$, let $A_i$ denote its $i^{th}$ row and $A^j$ denote its $j^{th}$ column; as always $A_{ij}$ denotes the $(i,j)^{th}$ entry.
(a) For \( m \times n \) and \( n \times p \) matrices \( A \) and \( B \) respectively, their product is an \( m \times p \) matrix \( C = AB \) where \( C_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} = A_iB^j \). Show that

\[
C = \sum_{k=1}^{n} (A^kB_k).
\]

Observe that each term on the right is the product of a \( m \times 1 \) column vector with a \( 1 \times p \) row vector to produce an \( m \times p \) matrix.

(b) There are at least two ways that one can represent a subspace \( W \) in \( \mathbb{R}^d \). The first is by a set of generators: We say that the vectors \( P_1, \ldots, P_k \in \mathbb{R}^d \) are generators for the subspace if \( W = \{ \alpha_1 P_1 + \cdots + \alpha_k P_k \mid \alpha_i \in \mathbb{R} \} \). A second way to represent the subspace \( W \) is by a set of constraints: Let \( A \in \mathbb{R}^{n \times d} \) be matrix. We say that \( A \) is a constraint matrix for the space \( W \) if

\[
W = \{ x \in \mathbb{R}^d \mid Ax = 0 \}
\]

i. Let \( W \) be a subspace of \( \mathbb{R}^d \) given by generators \( P_1, \ldots, P_k \in \mathbb{R}^d \). Explain how to write \( W \) via a constraint matrix \( A \).

ii. Let \( W \) be a subspace of \( \mathbb{R}^d \) given by a constraint matrix \( A \). Explain how to write \( W \) via a set of generators.

Problems

1. **Asymptotic Notation** (20 pts)

   For each list of functions, order them according to increasing asymptotic growth. Provide a brief argument justifying each successive step in the ordering.

   List 1, fast growing functions: \( 2^{3n}, 3^{2n}, n!, n^{\lg n}, n^n\sqrt{\lg n} \).

   List 2, slow growing functions: \( 2^{\lg n}, \lg(3^n), (\lg n)^5, \lg \lg \sqrt{n}, \lg(n^5) \).

   In this course, we use \( \lg = \log_2 \) (i.e., logarithm with base 2) and \( \ln = \log_e \). When we say \( \log \) (and do not specify the base), it is usually when we care only about the asymptotics and the base does not matter: it can be set to any constant.

2. **Asymptotic Estimates** (20 pts)

   Your solutions for this problem should begin with a displayed equation of the following form:

   \[
   \text{[given expression to be analyzed]} = \Theta(f(n)),
   \]

   where \( f(n) \) is a “simple” function. For example, the expression for the Harmonic number \( \sum_{i=1}^{n} (1/i) = \Theta(\log(n)) \). (Prove this for yourself, say using the hint of part (b).) You should then give an explanation of how you derived your answer (but a completely formal proof is not required).

   (a) Asymptotically simplify

   \[
   \sqrt{n+1} - \sqrt{n}.
   \]

   (b) Asymptotically simplify

   \[
   \sum_{i=1}^{n} i^2.
   \]

   (**Hint:** Try relating the discrete sum to an integral.)
(c) Asymptotically simplify
\[ \sum_{i=n+1}^{n^2} \frac{1}{i} \]

3. Recurrences (20 pts)
Solve the following recurrences, giving your answer in \( \Theta \) notation. For each of them, assume the base case \( T(x) = 1 \) for \( x \leq 5 \). Show your work.

a) \( T(n) = 3T(n/4) + n \).

b) \( T(n) = T(n-2) + n^4 \).

c) \( T(n) = 2T(n-5) \).

d) \( T(n) = \sqrt{n}T(\sqrt{n}) + n \).

e) \( T(n) = 2T(\sqrt{n}) + \log n \).

4. Linear Algebra (10 pts)
Given a unit vector \( u \in \mathbb{R}^d \), show that the orthogonal projection of another vector \( v \in \mathbb{R}^d \) onto the one-dimensional subspace spanned by \( u \) is given by \( uu^\top v \). Consider a matrix \( U \in \mathbb{R}^{d \times k} \) whose columns are orthonormal vectors \( u_1, u_2, \ldots, u_k \). Show that the projection of some vector \( v \in \mathbb{R}^d \) onto the column span of \( U \) is given by \(UU^\top v \).

5. Greedy Algorithms (15 pts)
You are at a conference and would like to attend as many talks as possible. There are \( n \) talks given and each has a starting time \( s_i \) and ending time \( e_i \). If you attend a talk you must sit through the entire talk (from start to finish). Naturally, you can only attend one talk at a time (i.e., you can’t attend a pair of talks if their intervals intersect). You are interested in finding the maximum number of talks you can attend.

a) Find an incorrect greedy algorithm that tries to solve this problem and give a counterexample explaining why it fails.

b) Find a correct greedy algorithm that solves this problem and give an argument why it always works.

6. Painting Windows (15 pts)
You are drawing a collection of \( n \) rectangular windows on your computer display which has size \( \ell \times h \). For each \( i \in \{1, \ldots, n\} \), the window \( w_i \) is given by a rectangle \( R_i \), a depth \( d_i \in \mathbb{Z} \) and a color \( c_i \in \mathbb{Z}_{\geq 0} \). Assume the depths are all distinct; windows with smaller depths hide windows with larger depths, if they overlap. You use the following algorithm:

\[
\text{for } i = 1 \text{ to } n:\n\quad \text{for each pixel in } R_i:\n\quad\quad \text{if undefined(pixel.current_depth) or pixel.current_depth > d_i:}\n\quad\quad\quad \text{set pixel.color = c_i and pixel.current_depth = d_i} \quad (\ast)
\]

Fix a pixel \( p \), and let \( N \) be the number of windows containing \( p \). Show that if the rectangles are permuted in a uniformly random order, the expected number of times (\( \ast \)) is executed for pixel \( p \) is \( O(\log N) \).