Lec 28: Random Walks and the Spectral Connection

\[ G \rightarrow A \rightarrow D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \\ \vdots & \ddots \end{pmatrix} \]

\[ L = D - A \]

\[ d_i = \text{degree of vertex } i \]

\[ 0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \]

\[ G \text{ undirected Unweighted} \]

\[ t = \text{exit} \]

\[ X_t = \text{position of process at } t \]

\[ X_{tn} = \text{uniformly random neighbor of } X_t \]

- \( o_1 \): cover time
- \( o_2 \): hitting time to target
- \( o_3 \): long-term density drift of \( X_t \)
- \( o_4 \): Why do I care?
0. $G =$ undirected / web graph
0. $G =$ Set all makeys / mbkp [G]

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01. Cover time = expected time to see all vertices $G$
02. Hitting time (start $s$, target $t$) = expected time to hit $t$ & start at $s$. $\approx O(mn)$
03. Limit distribution of random walk process $\mathbb{E}[X_T | X_0 = s] = \frac{du}{dx}$

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[Diagram with nodes and edges labeled MCMC]
\[ b^{(t)} = (b_1^{(t)}, \ldots, b_n^{(t)}) \subseteq [0,1]^n \quad \sum_i b_i^{(t)} = 1 \]

\[ \text{Claim 1:} \quad \sum_j b_j^{(t)} = \frac{d_i}{\sum_j d_j} = \frac{d_i}{2m} \]

\[ \implies b_i^{(t_n)} = \frac{d_i}{2m} \]

\[ b_i^{(t_n)} = \mathbb{P}[X_{t_n} = i] = \sum_j \mathbb{P}[X_{t_n} = i \mid X_t = j] \mathbb{P}[X_t = j] \]

\[ = \sum_j \left\{ \begin{array}{ll} \frac{d_j}{d_i} & \text{if } j \neq i \\ 0 & \text{otherwise} \end{array} \right\} \frac{d_j}{2m} \]

\[ = \sum_i \frac{d_i}{2m} \cdot \frac{1}{d_i} = \frac{d_i}{2m} \]
$p^{(0)}$ as a row vector

Claim: $p^{(t+1)} = p(t) \cdot D^{-1} A$

Proof by algebra

$\Rightarrow$ if $p$ is fixed point

Then $p = p \cdot D^{-1} A \Rightarrow p \cdot (1 - D^{-1} A) = 0$

$\Leftrightarrow p \cdot D^{-1} (D - A) = 0$

$\Leftrightarrow p \cdot D^{-1}$ is an eigenvector of $L$

$\Rightarrow p \cdot D^{-1} = \lambda p \Rightarrow p \propto A \lambda$
\[ |\phi_0\rangle = (1, 0, 0, \ldots, 0) \]

\[ |\phi_0\rangle \rightarrow |\phi_1\rangle = |\phi_0\rangle D - A |\phi_0\rangle M \]

\[ |\phi_0\rangle \rightarrow |\phi_0\rangle = |\phi_0\rangle M = |\phi_0\rangle M^2 \ldots \]

\[ |\phi_0\rangle \rightarrow |\phi_0\rangle M^2 \rightarrow |\phi_0\rangle M^3 \rightarrow |\phi_0\rangle M^4 \rightarrow \ldots \]

\[ \text{powers iteration} \]

\[ \Rightarrow \text{can be used to show that} \]

\[ \text{if } G \text{ is connected then and non-bip} \]

\[ \sum_{i, j} |p_t^i - p_t^j| \leq \varepsilon \text{ after } t = \text{poly}(m, \log \frac{1}{\varepsilon}) \text{ steps} \]
\[(Lx)_i = ((D-A)x)_i = (Dx-Ax)_i \]

\[= d_i x_i - \sum_{j \in \mathcal{N}_i} x_j \]

\[= d_i \left[ x_i - \frac{1}{d_i} \sum_{j \in \mathcal{N}_i} x_j \right] \]

\[\text{average of its neighbors} \]

\[\text{value} \]

\[\text{my value} \]

Laplacians arise in "averaging" type processes.
Sp. I am at $v$. What is the $P_0$ [hit $t$ before $s$]

$$P_0 = 1$$

$$P_1 = 0$$

Boundary conditions:

$$u \neq s, \tau$$

$$P_u = \frac{1}{2} (P_{u+1}) + \frac{1}{2} P_{u-1}$$

$$\frac{1}{2} \sum_{y \neq u} P_y$$

$$V = IR$$

$$\phi_a - \phi_b = \frac{f a b}{x R a b}$$

$\phi$ flow around $\Omega$, $u \neq s, t$