Outline

- Linear Programming Duality
- Application to zero sum games
\[ P = \max (2x_1 + 3x_2) \]
\[
\text{s.t.} \quad 4x_1 + 8x_2 \leq 12 \\
2x_1 + x_2 \leq 3 \\
3x_1 + 2x_2 \leq 4 \\
x_1, x_2 \geq 0
\]

Since \(2x_1 + 3x_2 \leq 4x_1 + 8x_2 \leq 12\), we know \(\text{OPT} \leq 12\)

Since \(2x_1 + 3x_2 \leq \frac{1}{2} (4x_1 + 8x_2) \leq 6\), we know \(\text{OPT} \leq 6\)

Since \(2x_1 + 3x_2 \leq \frac{1}{3} ((4x_1 + 8x_2) + (2x_1 + x_2)) \leq 5\), we know \(\text{OPT} \leq 5\)
Duality

• We took non-negative linear combinations of the constraints
• How do we find the best upper bound on OPT this way?
• Let $y_1, y_2, y_3 \geq 0$ be the coefficients of our linear combination. Then,

\[
\begin{align*}
4y_1 + 2y_2 + 3y_3 &\geq 2 \\
8y_1 + y_2 + 2y_3 &\geq 3 \\
y_1, y_2, y_3 &\geq 0
\end{align*}
\]

and we seek $\min(12y_1 + 3y_2 + 4y_3)$

\[
\begin{align*}
P &= \max(2x_1 + 3x_2) \\
\text{s.t.} & \quad 4x_1 + 8x_2 \leq 12 \\
& \quad 2x_1 + x_2 \leq 3 \\
& \quad 3x_1 + 2x_2 \leq 4 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]
Primal LP

\[ P = \max(2x_1 + 3x_2) \]
\[ \text{s.t. } 4x_1 + 8x_2 \leq 12 \]
\[ 2x_1 + x_2 \leq 3 \]
\[ 3x_1 + 2x_2 \leq 4 \]
\[ x_1, x_2 \geq 0 \]

Dual LP

\[ 4y_1 + 2y_2 + 3y_3 \geq 2 \]
\[ 8y_1 + y_2 + 2y_3 \geq 3 \]
\[ y_1, y_2, y_3 \geq 0 \]

and we seek \( \min(12y_1 + 3y_2 + 4y_3) \)

- If \((x_1, x_2)\) is feasible for the primal, and \((y_1, y_2, y_3)\) feasible for the dual,
  \[ 2x_1 + 3x_2 \leq 12y_1 + 3y_2 + 4y_3 \]
- If these are equal, we’ve found the optimal value for both LPs
- \((x_1, x_2) = \left(\frac{1}{2}, \frac{5}{4}\right)\) and \((y_1, y_2, y_3) = \left(\frac{5}{16}, 0, \frac{1}{4}\right)\) give the same value 4.75, so optimal
Dual LP

\[
\begin{align*}
4y_1 + 2y_2 + 3y_3 & \geq 2 \\
8y_1 + y_2 + 2y_3 & \geq 3 \\
y_1, y_2, y_3 & \geq 0
\end{align*}
\]

and we seek \( \min(12y_1 + 3y_2 + 4y_3) \)

• Let’s try do the same thing to the dual:
• \( 12y_1 + 3y_2 + 4y_3 \geq 4y_1 + 2y_2 + 3y_2 \geq 2 \)
• \( 12y_1 + 3y_2 + 4y_3 \geq 8y_1 + y_2 + 2y_3 \geq 3 \)
• \( 12y_1 + 3y_2 + 4y_3 \geq \frac{2}{3} (4y_1 + 2y_2 + 3y_2) + (8y_1 + y_2 + 2y_3) \geq \frac{4}{3} + 3 \)
Dual LP

\[
\begin{align*}
4y_1 + 2y_2 + 3y_3 &\geq 2 \\
8y_1 + y_2 + 2y_3 &\geq 3 \\
y_1, y_2, y_3 &\geq 0 \\
\text{and we seek } \min(12y_1 + 3y_2 + 4y_3)
\end{align*}
\]

\[P = \max(2x_1 + 3x_2)
\]
\[
\begin{align*}
4x_1 + 8x_2 &\leq 12 \\
2x_1 + x_2 &\leq 3 \\
3x_1 + 2x_2 &\leq 4 \\
x_1, x_2 &\geq 0
\end{align*}
\]

• Take non-negative linear combination of the two constraints
• How do we find the best lower bound on OPT this way?
• Let \(x_1, x_2 \geq 0\) be the coefficients of our linear combination. Then,
  • \(4x_1 + 8x_2 \leq 12, \ 2x_1 + x_2 \leq 3, \ 3x_1 + 2x_2 \leq 4, \ x_1 \geq 0, \ x_2 \geq 0\)
    and we seek to maximize \(2x_1 + 3x_2\)

We got back the primal!
Non-Nice Constraints

\[ P = \max(7x_1 - x_2 + 5x_3) \]
\[ \text{s.t.} \quad x_1 + x_2 + 4x_3 \leq 8 \]
\[ 3x_1 - x_2 + 2x_3 \geq 3 \]
\[ x_1, x_2, x_3 \geq 0 \]

\[ D = \min(8y_1 + 3y_2) \]
\[ \text{s.t.} \quad y_1 + 3y_2 \geq 7 \]
\[ y_1 - y_2 \geq -1 \]
\[ 4y_1 + 2y_2 \geq 5 \]
\[ y_1 \geq 0, y_2 \leq 0 \]
Formal Definition of Duality

**Primal**
Max \( c^T x \)
subject to \( Ax \leq b \)
\( x \geq 0 \)

**Dual**
Min \( b^T y \)
subject to \( A^T y \geq c \)
\( y \geq 0 \)

• Dual of the dual is the primal!
• Can we get better upper/lower bounds by looking at more complicated combinations of the inequalities, not just linear combinations?
Weak Duality

<table>
<thead>
<tr>
<th>Primal</th>
<th>Dual</th>
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<tr>
<td>Max $c^T x$</td>
<td>Min $b^T y$</td>
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<tr>
<td>subject to $A x \leq b$</td>
<td>subject to $A^T y \geq c$</td>
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<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
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- (Weak Duality) If $x$ is a feasible solution of the primal, and $y$ is a feasible solution of the dual, then $c^T x \leq b^T y$

- Proof: Since $x \geq 0$ and $y \geq 0$,
  
  $c^T x \leq y^T A x \leq y^T b = b^T y$
Strong Duality

**Primal**

Max $c^T x$

subject to $Ax \leq b$

$x \geq 0$

**Dual**

Min $b^T y$

subject to $A^T y \geq c$

$y \geq 0$

• (Strong Duality) If primal is feasible and bounded (i.e., optimal value is not $\infty$), then dual is feasible and bounded. If $x^*$ is optimal solution to the primal, and $y^*$ is optimal solution to dual, then

$$c^T x^* = b^T y^*$$

• To prove $x^*$ is optimal, I can give you $y^*$ and you can check if $x^*$ is feasible for the primal, $y^*$ is feasible for the dual, and $c^T x^* = b^T y^*$
Consequences of Duality

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I means infeasible
O means feasible and bounded
U means unbounded

*Which combinations are possible?*
Consequences of Duality

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I means infeasible
O means feasible and bounded
U means unbounded

Check means possible
X means impossible
Possible Scenarios

• Suppose primal is feasible and bounded

• By strong duality, dual is feasible and bounded

• If primal (maximization) is unbounded, by weak duality, $c^T x \leq b^T y$, so no feasible dual solution
e.g., max $x_1$ subject to $x_1 \geq 1$ and $x_1 \geq 0$
dual will have $y_1 \leq 0$ and $y_1 \geq 1$

• Can primal and dual both be infeasible?
  • Primal: max $2x_1 - x_2$ subject to $x_1 - x_2 \leq 1$ and $-x_1 + x_2 \leq -2$ and $x_1 \geq 0, x_2 \geq 0$
  • Dual: $y_1 \geq 0$, $y_2 \geq 0$, and $y_1 - y_2 \geq 2$ and $-y_1 + y_2 \geq -1$, and min $y_1 - 2y_2$
  • Constraints are same for primal and dual, and both infeasible
Strong Duality Intuition

Suppose $x^*$ satisfies $a_1 x = b_1$ and $a_2 x = b_2$
Strong Duality Intuition

For non-negative \( y_1 \) and \( y_2 \)

\[
c = y_1 a_1 + y_2 a_2.
\]

\[
c^T \cdot x^* = (y_1 a_1 + y_2 a_2) \cdot x^*
\]

\[
= y_1 (a_1 \cdot x^*) + y_2 (a_2 \cdot x^*)
\]

\[
= y_1 b_1 + y_2 b_2
\]

Defining \( y = (y_1, y_2, 0, \ldots, 0) \), we get

optimal value of primal = \( c^T x^* = b^T y \) = value of dual solution \( y \).

the \( y \) we found satisfies \( c = y_1 a_1 + y_2 a_2 = \sum_i y_i a_i = A^T y \), and hence \( y \) satisfies the dual constraints \( y^T A \geq c^T \) by construction.

But \( b^T y \geq c^T x^* \) by weak duality, so \( y \) is optimal!
Duality in Zero-Sum Games

• R is an n x m row payoff matrix
• W.l.o.g. R has all non-negative entries
• Variables: \( v, p_1, \ldots, p_n \)
• Max \( v \)
  subject to \( p_i \geq 0 \) for all rows \( i \), \( \sum_i p_i = 1 \), \( \sum_i p_i R_{i,j} \geq v \) for all columns \( j \)

• Replace \( \sum_i p_i = 1 \) with \( \sum_i p_i \leq 1 \).
• Include \( v \geq 0 \)
• Write \( \sum_i p_i R_{i,j} \geq v \) as \( v - \sum_i p_i R_{i,j} \leq 0 \)
Duality in Zero-Sum Games

\[
\max c^T x \text{ subject to } Ax \leq b \text{ and } x \geq 0
\]

\[
x = \begin{bmatrix} v \\ p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix}, \text{ and } A = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}
\]

• Dual: \( \min y^T b \text{ subject to } y^T A \geq c^T \text{ and } y \geq 0 \) for \( y = (y_1, \ldots, y_{m+1}) \)

• Dual constraints say \( y_1 + \cdots + y_m \geq 1 \) and \( \sum_j y_j R_{ij} \leq y_{m+1} \) for all rows \( i \)
  • Since we’re minimizing \( y_{m+1} \) and \( R_{ij} \) all non-negative, \( y_1 + \cdots + y_m = 1 \)

• \( y_{m+1} \) is value to the row player and \( y_1, \ldots, y_m \) is column player’s strategy

• **Strong duality:** \( \max \min \sum_i p_i R_{ij} = \min \max \sum_j y_j R_{ij} \)