Taking Duals I. Consider this maximization linear program:

\[
\begin{align*}
\text{max} & \quad x_1 + 3x_2 - 2x_3 \\
\text{s.t.} & \quad x_1 + x_2 + 2x_3 \leq 2 \\
& \quad 7x_1 + 2x_2 + 5x_3 \leq 6 \\
& \quad 2x_1 + x_2 - x_3 \leq 1 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

1. Write down its dual LP. (Is it a maximization or minimization problem? What are the variables? Constraints?)

**Solution:** Remember, we’re trying to find the best (i.e., lowest) upper bound on the value of the primal LP. So it should be a *minimization*. There’s one variable \( y_i \) for each constraint in the primal, so that’s three \( y_i \) variables. We’re summing up constraints of the form \( \text{blah} \geq \text{blah} \), so we want each \( y_i \geq 0 \) to not flip the inequalities. Etc.

\[
\begin{align*}
\text{min} & \quad 2y_1 + 6y_2 + 1y_3 \\
\text{s.t.} & \quad y_1 + 7y_2 + 2y_3 \geq 1 \\
& \quad y_1 + 2y_2 + y_3 \geq 3 \\
& \quad 2y_1 + 5y_2 + (-1)y_3 \geq -2 \\
& \quad y_1, y_2, y_3 \geq 0
\end{align*}
\]

2. Now write down the dual of this dual LP. Remember, since the dual will be a minimization LP, this dual’s-dual will give a best lower bound on the dual.

**Solution:** The dual’s dual will be the same as the primal. Details are omitted here.

Taking Duals II. Write down the dual of this minimization LP: *(Be careful, some inequalities are greater-than, some are less-than. And not all constraints have all variables.)*

\[
\begin{align*}
\text{max} & \quad x_1 - 3x_2 + 2x_3 \\
\text{s.t.} & \quad 3x_1 + 2x_3 \geq 2 \\
& \quad 2x_2 - x_3 \leq 5 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

**Solution:** To make life easy, first convert this into a nicer form. Since the dual wants to give a *upper bound* on this *maximization* problem, let’s make the constraints of the form \( \text{blah} \leq \text{blah} \).

\[
\begin{align*}
\text{max} & \quad x_1 - 3x_2 + 2x_3 \\
\text{s.t.} & \quad -3x_1 - 2x_3 \leq -2 \\
& \quad 2x_2 - x_3 \leq 5 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]
And not all variables appear in all constraints, so let’s put down zeros where things are missing.

\[
\begin{align*}
\text{max} & \quad x_1 - 3x_2 + 2x_3 \\
\text{s.t.} & \quad -3x_1 + 0x_2 - 2x_3 - 2 \\
& \quad 0x_1 + 2x_2 - x_3 \leq 5 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

Finally, use a dual variable \( y_i \geq 0 \) for each primal constraint, and two constraints, etc.

\[
\begin{align*}
\text{min} & \quad -2y_1 + 5y_2 \\
\text{s.t.} & \quad -3y_1 + 0y_2 \geq 1 \\
& \quad 0y_1 + 2y_2 \geq -3 \\
& \quad -2y_1 - y_2 \geq 2 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

C. Minimax from Duality (by Example). Let the row-player’s payoffs be given by this (non-negative) matrix

\[
\begin{array}{c|cc}
 & L & R \\
\hline
L & 1 & 5 \\
R & 3 & 2
\end{array}
\]

1. If the probabilities on the two rows are \( p_1 \) and \( p_2 \), write down an LP for the row player’s optimal strategy:

**Solution:** If the row player puts \( p_1 \geq 0 \) and \( p_2 \geq 0 \) on \( L, R \) respectively, then she wants to solve max \( p_1 + 3p_2, 5p_1 + 2p_2 \). I.e., the LP is

\[
\begin{align*}
\text{max} & \quad v \\
\text{subject to} & \quad p_1 + 3p_2 \geq v \\
& \quad 5p_1 + 2p_2 \geq v \\
& \quad p_1 + p_2 \leq 1 \\
& \quad p_1, p_2 \geq 0.
\end{align*}
\]

We set \( p_1 + p_2 \leq 1 \), but to maximize \( v \), the LP will automatically set the sum equal to 1.

2. Now take the dual of this LP. Show this dual is an LP computing the column player’s optimal strategy. (And hence strong duality implies the minimax theorem.)

**Solution:** First convert into inequalities \( \text{blah} \leq \text{blah} \) useful to show an upper bound. (Also, since all payoffs are non-negative, we can add in non-negativity for \( v \).)

\[
\begin{align*}
\text{max} & \quad v \\
\text{subject to} & \quad v - p_1 - 3p_2 \leq 0 \\
& \quad v - 5p_1 - 2p_2 \leq 0 \\
& \quad p_1 + p_2 \leq 1 \\
& \quad v, p_1, p_2 \geq 0.
\end{align*}
\]
If the dual variables are $q_1, q_2$ and $w$, we get

\[
\begin{align*}
\min w \\
\text{subject to} & \quad q_1 + q_2 \geq 1 \\
& \quad -q_1 - 5q_2 + w \geq 0 \\
& \quad -3q_1 - 2q_2 + w \geq 0 \\
& \quad w, q_1, q_2 \geq 0
\end{align*}
\]

Now move some variables around:

\[
\begin{align*}
\min w \\
\text{subject to} & \quad q_1 + q_2 \geq 1 \\
& \quad w \geq q_1 + 5q_2 \\
& \quad w \geq 3q_1 + 2q_2 \\
& \quad w, q_1, q_2 \geq 0
\end{align*}
\]

And again observe that to minimize the value, any optimal solution will reduce $q_1, q_2$ to make their sum equal to 1. So the LP is solving:

\[
\min_{q_1 + q_2 = 1, q_1, q_2 \geq 0} \max (q_1 + 5q_2, 3q_1 + 2q_2).
\]

That’s the column player’s strategy!!! And by strong duality, we get the minimax theorem for this particular game. Exactly the same idea holds in general, details are in the lecture notes.