Dynamic Programming

Off-Line Stock Market Problem You’re given a sequence of stock prices \([p_1, p_2, \ldots, p_n]\). You want to find the maximum profit that you could have made on the stock in hindsight. In other words, you want to find \(i\) and \(j\) with \(1 \leq i \leq j \leq n\) such that \(p_j - p_i\) is maximal. Your algorithm should run in \(O(n)\) time.

Solution: Scan the sequence from first to last. After processing \(p_i\) keep two things: (1) \(m_i\) the minimum of \(p_1, \ldots, p_i\), and (2) \(g_i\) the maximum profit achievable so far. It’s easy to update these when processing the next stock price \(p_{i+1}\).

\[
\begin{align*}
  m_{i+1} &= \min(m_i, p_{i+1}) \\
  g_{i+1} &= \max(g_i, p_{i+1} - m_{i+1})
\end{align*}
\]

The final answer is \(m_n\). (For completeness, note that \(m_0 = \infty\) and \(g_0 = -\infty\).)

Longest Increasing Subsequence: Given an array \(A\) of \(n\) integers like \([7, 2, 5, 3, 4, 6, 9]\), find the longest subsequence that’s in increasing order (in this case, it would be \(2, 3, 4, 6, 9\)). Give a dynamic-programming algorithm that runs in time \(O(n^2)\) to solve this problem.

1. To keep things simple, first let’s say you just need to output the *length* of the longest-increasing subsequence. E.g., in the above case, the length is 5.

   *Hint: suppose that for each \(i' < i\) you have computed the length of the LIS of \(A_0 \ldots i'\) that ends with \(A[i']\). How would you use this to solve the corresponding problem for \(i\)?

   **Solution:** \(L[i] = \max \{L[i'] + 1 : i' < i, A[i'] < A[i]\}\), or \(L[i] = 1\) if there are no such \(i'\).

2. Now extend your solution to actually find the LIS.

   **Solution:** One approach is when computing the max above, to also have a separate array that stores the argmax, that is, the index \(i'\) such that \(L[i] = L[i'] + 1\). One can then read off the sequence by going backwards from the end.

Closest Depot in a Tree: You’re given a rooted tree \(T\) with \(n\) vertices. There are \(m \leq n\) special vertices called *depots*. You are to compute, for every node \(v\) of \(T\), the distance from \(v\) to the nearest depot. The distance is the number of tree edges that must be traversed to get there. (If \(v\) is a depot the distance is 0.)

Your algorithm should work by doing one or two depth-first searches (DFS) of the tree starting from the root, and it should run in \(O(n)\) time.
Solution: First, compute for each node \( x \) the distance to the closest depot at or under it, denote this by \( U(x) \). For each leaf this is either 0 (if the leaf is a depot) or \( \infty \) (if it is not a depot). For every other node, \( U(x) \) is 0 (if \( x \) is a depot) or \( 1 + \min_{\text{child of } x} U(y) \) (if not).

Now let \( D(x) \) denote the distance to the closest depot to \( x \) in any direction. Clearly \( D(\text{root}) = U(\text{root}) \). Moreover, for any other \( x \) with parent \( p_x \), \( D(x) = \min(U(x), 1 + D(p_x)) \).

Making Change: You are given denominations \( v_1, v_2, \ldots, v_n \) (all integers) of the various kinds of currency you have. (Say \( v_1 = 1 \), so you can make change for any integer amount \( C \geq 1 \).) Given \( C \), give a dynamic programming solution which makes change for \( C \) with the fewest bills possible.

(Again, as a first stab, compute the number of bills required, and then extend the solution to output the number of bills of each denomination needed.)

Solution: Create an array \( B \) where \( B[C'] \) represents the fewest bills needed to make change for \( C' \). We can fill this in using the formula \( B[C'] = \min\{B[C'-v_i]+1 : v_i \leq C'\} \), where we begin with \( B[0] = 0 \) and then work upward from \( C' = 1 \) to \( C \). The total time taken is \( O(Cn) \).

Making Change (Part II): Now suppose you have only one bill of each denomination \( i \). Given \( C \), give a dynamic programming solution which makes change for \( C \) using the fewest bills, using no more than one bill of each denomination \( i \) (or says this is not possible).

Solution: One approach is to create a 2-dimensional array \( B \) where \( B[C', i] \) represents the fewest bills needed to make change for \( C' \) using denominations \( 1, 2, \ldots, i \) only (or infinity if it is not possible). Base case \( B[0, 0] = 0 \) and \( B[C', 0] = \infty \) for \( C' > 0 \). For general values of \( i \) we have \( B[C', i] = \min(B[C', i-1], B[C'-v_i, i-1] + 1) \) if \( C'-v_i \geq 0 \) or else \( B[C', i] = B[C', i-1] \) if \( C'-v_i < 0 \).

Making Change (Part III): Can you solve the problem if you have \( \ell_i \) bills of denomination \( i \)?

Solution: We can just modify the formula for \( B \) above to:

\[
B[C', i] = \min\{B[C' - jv_i, i - 1] + j : 0 \leq j \leq \ell_i, C' - jv_i \geq 0\}.
\]

Balanced Partition. You have a set of \( n \) integers each in the range \( 0, \ldots, K \). In time \( O(n^2K) \), partition these integers into two subsets such that you minimize \( |S_1 - S_2| \), where \( S_1 \) and \( S_2 \) denote the sums of the elements in each of the two subsets.

Solution: Let \( S \) be the sum of all the integers. Then \( S \leq nK \). To minimize \( |S_1 - S_2| \) it suffices to find a set \( A_1 \) whose numbers sum to \( S_1 \leq [S/2] \), that is as close to \( S/2 \) as possible. And this can be done by a dynamic program like for knapsack, in time \( O(nS) = O(n^2K) \).