the adversary game

part I: Claude’s Education Begins
Shall we play a game?

twenty questions...

I guess a number \( N \) between 1 and \( 2^{20} \). You ask yes/no questions of the form “is \( N \leq \text{blah} \)”, and after 20 questions you must tell me what \( N \) is.
sure!
Heh, this Claude guy sounds like an easy target...

I will not think of a single N, of course.

Each time I will try to answer in such a way that at the end, there are at least 2 numbers satisfying all the answers I have given.

So whatever number he says at the end, I can say N was the other number!

hee hee.
Initially, no answers given: N could be any number in $[1,2^{20}]$

OK Claude, ask away...

Is $N \leq 500K$?
Let's see. If I say "yes", N lies in [1, 500000],
else it lies in [500001, 2^{20} = 1048576].

I have more options if I go with "no"

No.

Is N <= 800K?
Hmm. If “yes”, N lies in [500001, 800000].
else N lies in [800000, 1048576].

Now I have more options if I go with “yes”.

... and so on ...

Yes.
after the 20th question...

The set of numbers \([561382, 583841]\)
all satisfy the answers I have given so far.

is it 575757?

No. Sorry, Claude, you lose, it was 575758!
rats!
Hmm, I see what he did there.
He did not really fix a single “true” N...

He answered so that
the “set of valid N’s” was at least
half of the previous set.

Since at least two valid Ns remained, I lost.
So my best bet is to halve the set of valid N's with each question!

(I messed up earlier: I did not halve each time.)

Let me call this “halving” search
Actually, "binary" search sounds better...
btw, can I win if I do binary search?

At the start, there are $2^{20}$ possible N's.
Each time I will exactly halve this range.
(and I can exactly halve each time)

So after 20 questions I can get the range down to a single value of N. And win.
Hmm. Also, 19 questions would not have been enough to find a number in $[1,2^{20}]$. (unless the adversary is silly, which he is not.) This sounds like a general principle.
ok, adversary, let's play again
Initially, no answers given:
N could be any number in $[1, 2^{20}]$

OK Claude, ask away...

Is $N \leq 2^{19} = 524288$?
If I say “yes”, N lies in $[1, 2^{19}]$, else it lies in $[2^{19}+1, 2^{20}]$.

I have the same # of options ($2^{19}$).
I can go with “no”.

No. $N > 2^{19}$.

Is $N \leq 2^{19} + 2^{18}$?
Hmm. If “yes”, $N$ lies in $[2^{19}, 2^{19}+2^{18}]$. Else $N$ lies in $[2^{19}+2^{18}, 2^{20}]$.

Equal number of options again. I can go with either answer. Let me choose one.

Yes.

... and so on ...
ok, adversary, after these 20 questions there's a unique number N consistent with your answers.

I win.

You just got lucky...
after a few games...
OK, ok, enough with these childish games, young Claude.

How about we play a real game?

tell me...
I think of integer N in \([1, 2^{20} + 1]\)...

But now you can ask me any twenty yes/no questions about N.

After 20 questions you must tell me what N is.

E.g., you can ask me:
- Is N in the set \([7, 9, 63, 256, 1079993]\)?
- or is \(2^{2^N} - 1\) a prime number?
- or is the 7th bit of N set?
- or are \((N, N+2)\) a Goldbach pair?
- whatever yes/no questions you want.
So I expand my range of N's by just one.

And give you the vast power to ask any possible yes/no questions

Should we play?
should Claude play?

why/why not?

Where will these games end?

stay tuned for the answers…
featuring

BSD “beastie” deamon
Claude Elwood Shannon

the adversary
Claude

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