## 15-451/651 Algorithm Design & Analysis, Spring 2024

# Extra Review Problems

## Linear Programming II

1. **(Short answer / multiple choice)**

   (a) Convert the following LP to standard form, and then write its dual.

$$\begin{aligned} \text{minimize } & -x_1 + 2x_2 \\ & x_1 \leq 9 \\ & 3x_2 + x_1 \geq 14 \end{aligned}$$

   (b) What is the dual of this linear program?

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 6x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 \leq 5 \\ & 6x_1 + 3x_2 + 3x_3 \leq 45 \\ & 2x_1 + x_2 + x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

   (c) What is the dual of this linear program?

$$\begin{aligned} \text{minimize} \quad & 2y_1 - 3y_2 \\ \text{s.t.} \quad & y_1 - 4y_2 \geq 6 \\ & 3y_1 + 2y_2 \leq 4 \\ & y_1, y_2 \geq 0 \end{aligned}$$

   (d) Suppose we have a primal LP with an objective coefficient vector $c$ and it's dual with objective coefficient vector $b$. We then find an assignment $x$ that satisfies the primal and an assignment $y$ that satisfies the dual. Select the strongest statement that must be true.

   - ☐ $c^T x = b^T y$
   - ☐ $c^T x \leq b^T y$
   - ☐ $c^T x \geq b^T y$

2. **(Maximum-flow, minimum-what?)** Let $G = (V, E)$ be a directed graph with edge capacities $c(u, v)$ for $(u, v) \in E$. Recall from lecture that the max-flow problem can be written as an LP. Defining a flow variable $f_{uv}$ representing $f(u, v)$ for every $(u, v) \in E$, we have the LP

$$
\begin{aligned}
\text{maximize} \quad & \sum_{(s,u)\in E} f_{su} - \sum_{(u,s)\in E} f_{us}. && \text{(the net } s\text{-}t \text{ flow)} \\
\text{s.t.} \quad & \sum_{\substack{v \text{ s.t.} \\ (v,u)\in E}} f_{vu} - \sum_{\substack{v \text{ s.t.} \\ (u,v)\in E}} f_{uv} = 0. && \text{for all } u \notin \{s, t\} \text{ (flow conservation)} \\
& 0 \le f_{uv} \le c(u, v) && \text{for all } (u, v) \in E \text{ (capacity constraints)}
\end{aligned}
$$

In standard form, this becomes

$$
\begin{aligned}
\text{maximize} \quad & \sum_{(s,u)\in E} f_{su} - \sum_{(u,s)\in E} f_{us}. && \text{(the net } s\text{-}t \text{ flow)} \\
\text{s.t.} \quad & \sum_{\substack{v \text{ s.t.} \\ (v,u)\in E}} f_{vu} - \sum_{\substack{v \text{ s.t.} \\ (u,v)\in E}} f_{uv} \le 0 && \text{for all } u \notin \{s, t\}, \text{ (flow conservation)} \\
& \sum_{\substack{v \text{ s.t.} \\ (u,v)\in E}} f_{uv} - \sum_{\substack{v \text{ s.t.} \\ (v,u)\in E}} f_{vu} \le 0 && \text{for all } u \notin \{s, t\}, \text{ (flow conservation)} \\
& f_{uv} \le c(u, v) && \text{for all } (u, v) \in E, \text{ (capacity constraints)} \\
& f_{uv} \ge 0 && \text{for all } (u, v) \in E, \text{ (capacity constraints)}
\end{aligned}
$$

Now that this LP is in standard form, we can apply our usual rules to take the dual.

(a) What is the dual of this problem?

(b) Simplify the dual LP as much as possible, which will involve making it *not* standard form. **Hints:**

- Remember that when we learned how to convert arbitrary LPs into standard form, we made substitutions like switching unbounded variables $x$ with two non-negative variables $x^+ - x^-$. You might be able to simplify the resulting LP by doing the opposite.
- $s$ and $t$ are special cases (they don't have conservation constraints) which means they will also end up as special cases in the dual. Creating some extra variables for them will help to eliminate redundant constraints.

(c) Intuit that this corresponds to min-cut. If you assume you get an integer solution to this LP, describe what the variables and their values represent, and how each constraint forces the solution to be a minimum cut.