15-451/651 Algorithm Design & Analysis, Spring 2024

Extra Review Problems

Algorithm Analysis, Linear-time Selection

1. **(Multiple choice)**

   (i) For $c > 1$, consider the summation $S = c + c^2 + c^3 + \ldots + c^n$. Which of the following are true? (Multiple may be true)

   (a) $S = \Omega(nc^n)$

   (b) $S = \Omega(nc)$

   (c) $S = O(c^n)$

   (d) $S = O(nc)$

   (e) $S = \Theta(c^n)$

   (f) $S = \Theta(nc^n)$

   (g) $S = \Theta(nc)$

   (ii) In the `DeterministicSelect` algorithm, the median of medians procedure

   (a) recursively calls itself

   (b) calls `DeterministicSelect`

   (c) Both of the above

   (iii) What would be the worst-case running time of the `Quicksort` algorithm if we pick an arbitrary (fixed) element as the pivot?

   (a) $\Theta(n)$

   (b) $\Theta(n \log n)$

   (c) $\Theta(n^2)$

   (d) $\Theta(n^2 \log n)$

   (iv) What would be the **expected** worst-case running time of the `Quicksort` algorithm if we pick a random element as the pivot?

   (a) $\Theta(n)$

   (b) $\Theta(n \log n)$

   (c) $\Theta(n^2)$

   (d) $\Theta(n^2 \log n)$

2. **(Recurrences)** Solve the following recurrences in Θ notation by the tree/brick method (assume all base cases have $T(1) = 0$):

   (a) $T(n) = 1 + 2T(n-1)$.

   (b) $T(n) = n + T(n/2)$.

   (c) $T(n) = n + 2T(n/2)$.

3. **(Probability Facts)** Let's follow the convention of using uppercase letters X, Y, Z to denote *random variables* (which we often abbreviate to *r.v.*s).

   Independence Random variables X, Y are *independent* if for any values a, b, we have

$$\Pr[X = a, Y = b] = \Pr[X = a] \cdot \Pr[Y = b].$$

   (a) Consider flipping two fair coins. Let $X \in \{0, 1\}$ be the outcome of the first coin (where we think of heads as 1 and tails as 0) and let $Y \in \{0, 1\}$ be the outcome of the second coin. Let $Z = X + Y \bmod 2$. Are X and Y independent? What about X and Z?

   Linearity of expectation Given any two r.v.s X, Y,

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

   This is true even if X and Y are not independent!

   (b) Suppose we take n books numbered 1 to n, and place them on a shelf in a (uniformly) *random* order. What is the expected number of books that end up in their correct location? ("correct" = location it would be in if they were sorted).

   Markov's inequality given a *non-negative random variable* X with mean/expectation $\mu = \mathbb{E}[X]$,

$$\Pr[X > c \cdot \mu] \leq \frac{1}{c}.$$

   (c) Show that if X is allowed to take negative values then the above inequality is no longer true.