15-451/651 Algorithm Design & Analysis, Spring 2024

Recitation #13

Objectives

- Provide practice reducing problems to properties and applications of polynomials.
- Practice applying online learning and multiplicative weights.

Recitation Problems

1. **(2-SUM and 3-SUM)**

   You are given an array of n integers A, each element of which is at most $O(n)$ in size. Your goal is to determine the possible sums that you can make by taking integers from this array.

   (a) Solve the 2-SUM problem. That is, output a list of all of the integers that you can possibly make by summing any pair of elements from A. You are allowed to use the same element twice.

   (b) Now solve 2-SUM but without allowing the same element to be used twice.

   (c) Solve the 3-SUM problem, That is, output a list of all of the integers that you can possibly make by summing any triple of elements from A. You are allowed to use the same element multiple times.

   (d) Finally, solve 3-SUM but without allowing the same element to be used multiple times.

2. **(Playing games like an expert)** A cool application of the randomized weighted majority algorithm is that it can be used to play two-player zero-sum games! Suppose we have a two-player zero-sum games where all of the payoffs are zero (column player wins) or one (row player wins). We will play as the column player, and use the experts framework to help us pick our strategy. Specifically, we will treat each column as an expert and use the randomized weighted majority algorithm. Each column (expert) begins with a weight of 1.

   - According to the randomized weighted majority algorithm, each round, we should play column i with probability

     $$\boxed{}$$

   - We should reduce the weights of ______________________________

     ______________________________ by ____________________

   - How well does this strategy perform in the long run? Lets appeal to the theorem that we proved in lecture, which says that

     $$\text{our error rate} \leq \text{optimal error rate} + 2\sqrt{\frac{\ln n}{T}}$$

     How should we interpret "our error rate" and "optimal error rate" in game-theoretic terms in relation to this problem? (Hint: first think about what a "mistake" corresponds to)

     Our error rate: ______________________________

     Optimal error rate: ______________________________

- Lastly, let's see how close this strategy is to optimal. Suppose that over the T rounds of the game, the row player played row i some portion p_i of the time (in other words, they played row i with probability p_i). Write the "optimal error rate" quantity from the previous part in terms of p. Using facts that we have learned about zero-sum games, what is an upper bound on this quantity that relates it to the game?

- Use the error rate bound for multiplicative weights that we wrote above and combine it with the upper bound from the previous part to show how good our strategy was

$$\underbrace{\rule{4cm}{0.4pt}}_{\text{"Our error rate"}} \leq \underbrace{\rule{4cm}{0.4pt}}_{\text{"Optimal error rate"}} + 2\sqrt{\frac{\ln n}{T}}$$

- What happens as $T \to \infty$?