15451 Fall 2022

Approximation Algorithms

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What can we do for NPC problems?

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Design poly-time approximation algorithms

Consider the solution version (rather than the decision version)

m machines, n jobs, j-th job takes time p_j

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partition jobs to the machines to minimize the *makespan*, defined as

$$\max_{i \in [m]} \left(\sum_{j \in S_i} p_j \right)$$

where S_i is the set of jobs assigned to machine i

Claim:

The job assignment problem is NPC

subset sum: x_1 . . . . . . x_n

does there exist a subset that sum up to k?

$$k = \frac{1}{2} \sum_{i=1}^{n} x_i$$

Greedy:
Take any unassigned job, give it to the machine with current minimum load

Claim:

Greedy achieves 2-approximation, i.e., achieves a makespan at most 2 times the optimal

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at the moment j^* is added, every machine has load at least L

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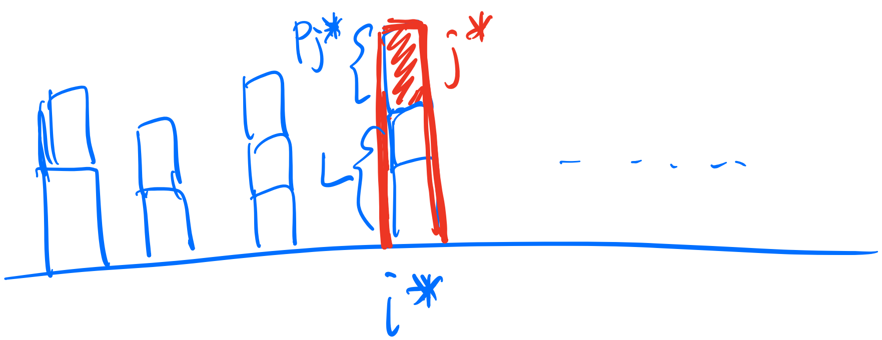
Let i^* be the most loaded machine, let j^* be the last job assigned to it



$$p_{j^*} + L \le 2 \cdot OPT$$

optimal

$$p_{j^*} \le OPT$$

$$L \le OPT$$

Can we do better?

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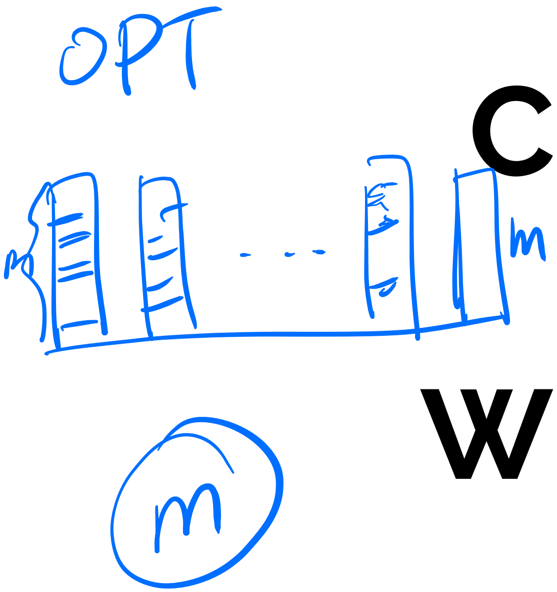
What is a bad case?



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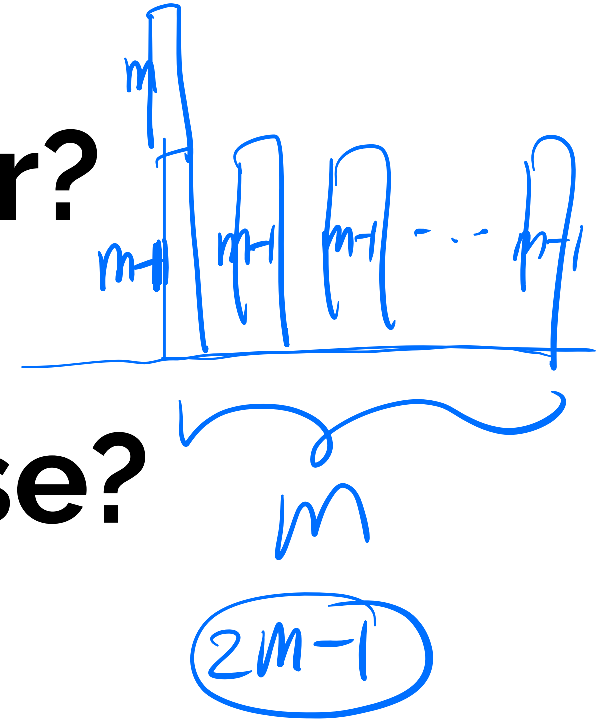

What is a bad case?

m(m-1) jobs of size 1
1 job of size m

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1 job of size m

Sorted Greedy:

Run greedy largest unassigned job first

1.5 approximation

$p_{j^*} + L \le 1.5\ OPT$

$L \le OPT$

$p_{j^*} \le 0.5\ OPT$

Let i* be the most loaded machine, let j* be the last job assigned to it

Want to show:

$$p_{j^*} + L \leq 1.5 \cdot OPT$$

Suffices to show: $p_{j^*} \leq OPT/2$

We may assume $L > 0$ (why?)

$L=0$

Each machine has at least one job assigned before, and its size is at least p_{j^*}

$OPT \geq 2p_{j^*}$

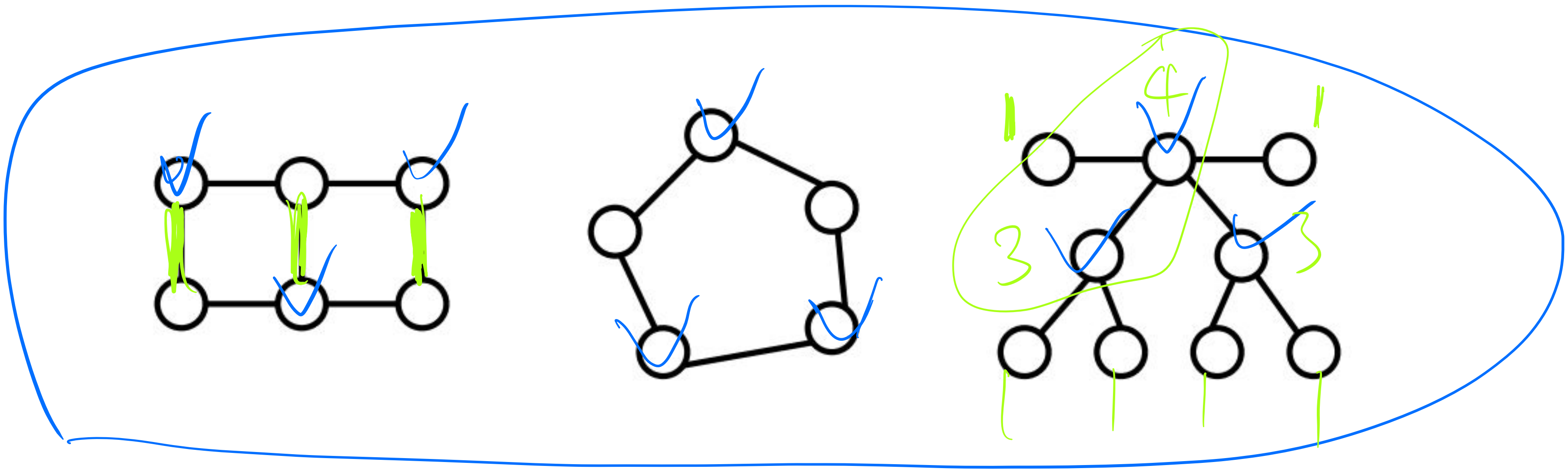
We may assume L > 0 (why?)

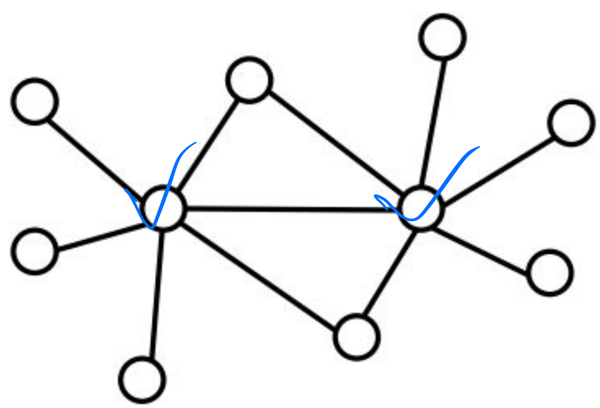
Each machine has at least one job assigned before, and its size is at least p_{j^*}

$$2p_{j^*} \leq OPT$$

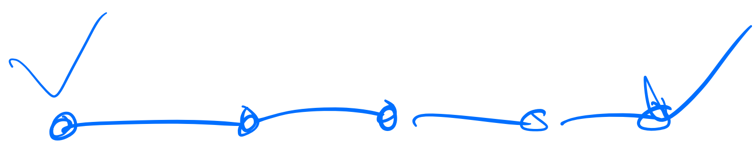
Vertex-Cover:
Given a graph G, find the smallest set of vertices such that every edge is incident to at least one of them.

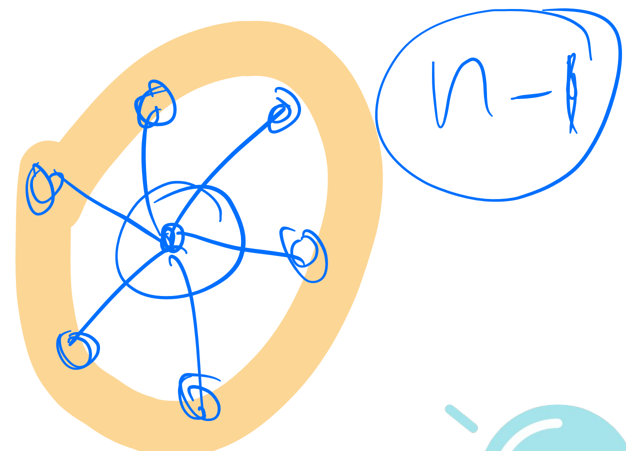
Decision problem: Given G and k, does G contain a vertex cover of size ≤ k?





Pick an arbitrary vertex with at least one uncovered edge incident to it, put it into the cover, and repeat.





Pick an arbitrary vertex with at least one uncovered edge incident to it, put it into the cover, and repeat.

Does this give a good approximation?

Pick the vertex that covers the most uncovered edges

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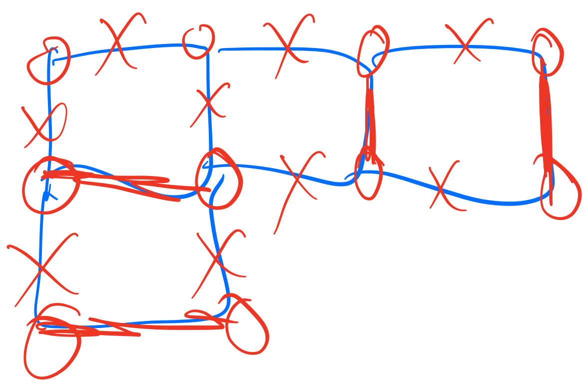
Achieves only O(log n) approximation

A 2-approximation algorithm

Pick an arbitrary edge. Add both endpoints.
Throw out all edges covered and repeat.
Continue until no uncovered edges left.



Proof: matching M ! ! !

$$|M| \leq OPT$$

Alg $\quad 2 \cdot |M| \leq 2OPT$

Another 2-approximation algorithm

"LP relaxation"

- **Find an LP solution (possibly fractional)**

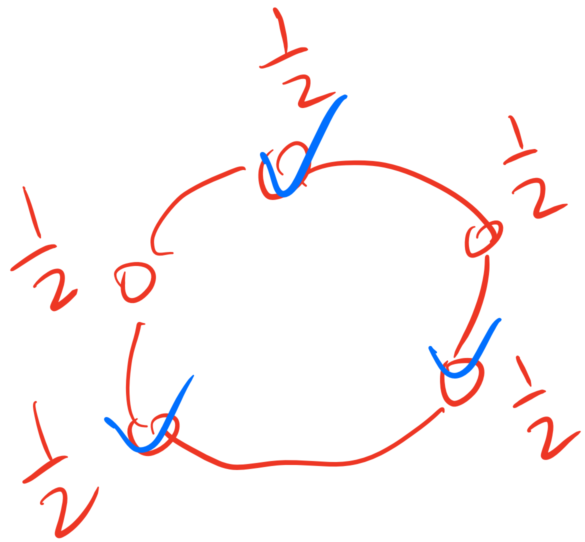
  $0 \le x_i \le 1$, for each edge (i, j), $x_i + x_j \ge 1$

  Minimize $\Sigma_i x_i$

- **Round the LP solution:**

  If $x_i \ge \frac{1}{2}$, set to 1, else set to 0

rounded solution $\le 2 \cdot LP \le 2 \cdot OPT$



LP = 2.5

OPT = 3

Known Results

Best known approximation:

$$2 - \Theta(1/\sqrt{\log n})$$

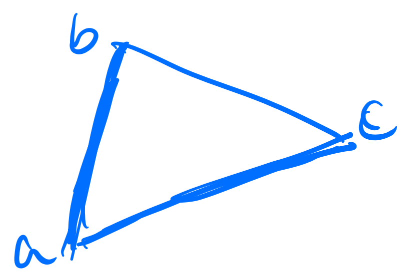
7/6 approximation is NP hard [Hastad]

Improved to 1.361 [Dinur and Safra]

Metric Traveling Salesman Problem

Find the shortest path to visit n cities, each exactly once, returning to where you started.

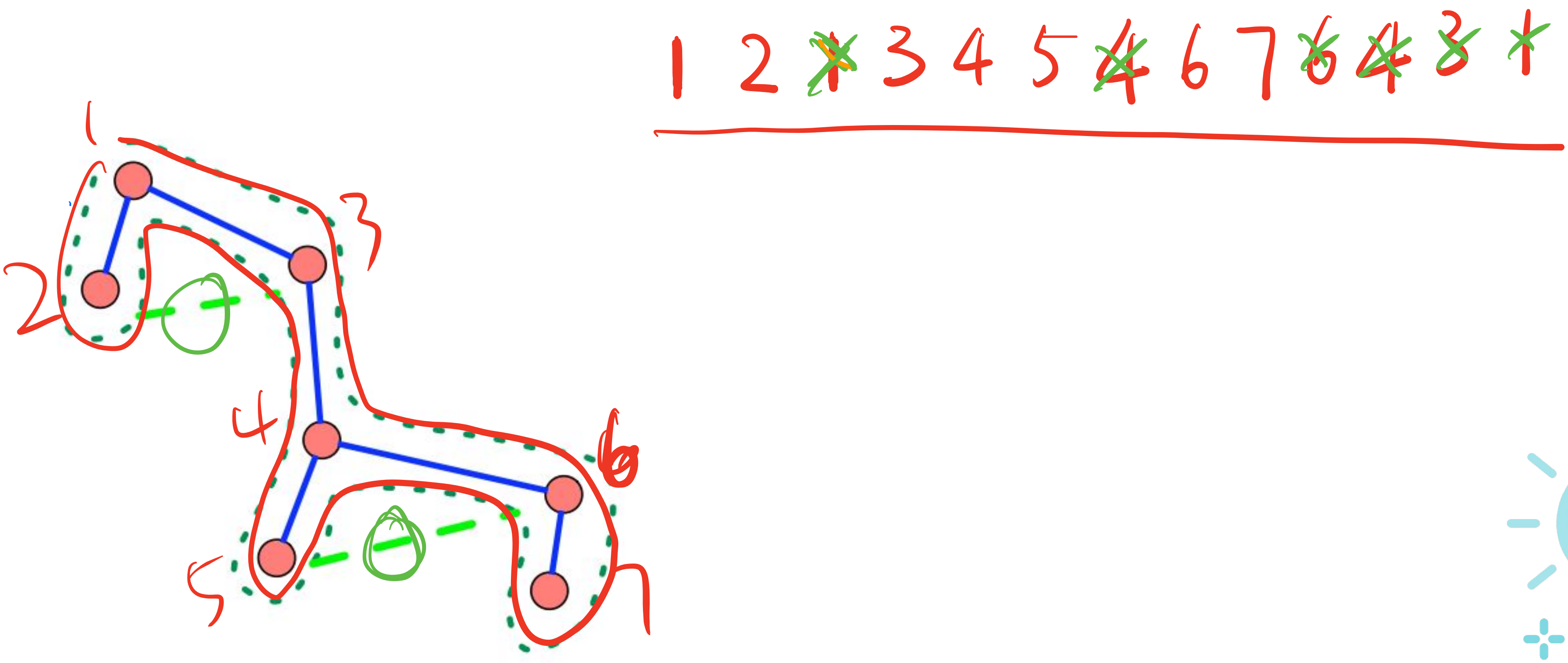
Metric: distances are symmetric and obey triangle inequality



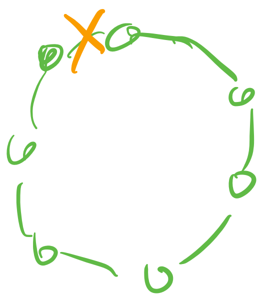
$d_{ab} + d_{bc} \geq d_{ac}$

Compute MST

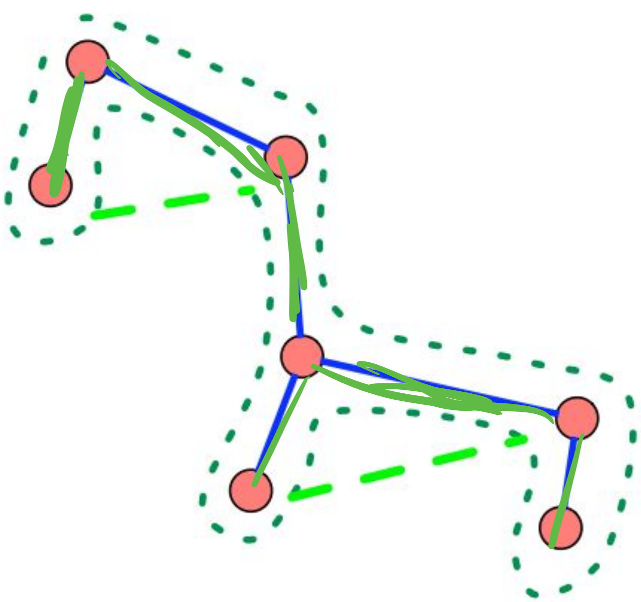
Output a pre-order traversal of the MST



This algorithm achieves 2 approximation.



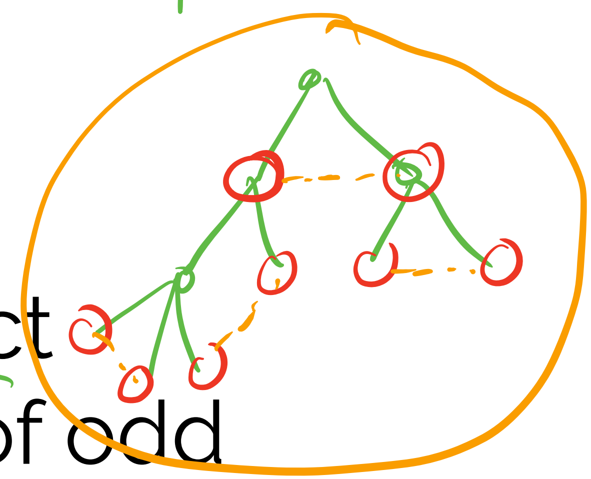
$|MST| \leq OPT$



$Alg \leq 2|MST| \leq 2\,OPT$

Christofide's algorithm

1.5 approx



Compute the MST T.

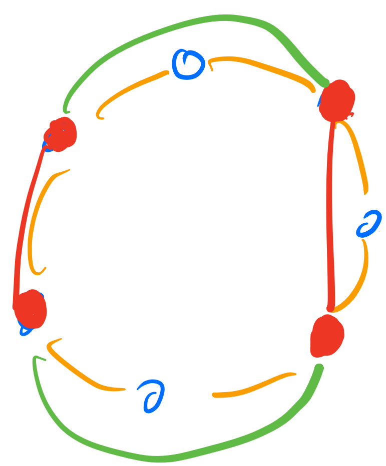
Compute a minimum weight perfect matching M between the vertices of odd degree in T.

G = T ∪ M. Return the TSP tour constructed from shortcutting an Eulerian tour on G.

Proof: $|T| + |M| \leq 1.5\ OPT$

$|T| \leq OPT$

suffices to show that $|M| \leq 0.5\ OPT$



●: odd degree vertices in T

$2|M| \leq Green + Red \leq OPT$

$|M| \leq Green \quad |M| \leq Red$

Karlin, Klein, and Oveis Gharan improves it to 1.4999999999999999999999999999999999 approximation.

(Not an April fools joke.
See https://arxiv.org/abs/2007.01409.)