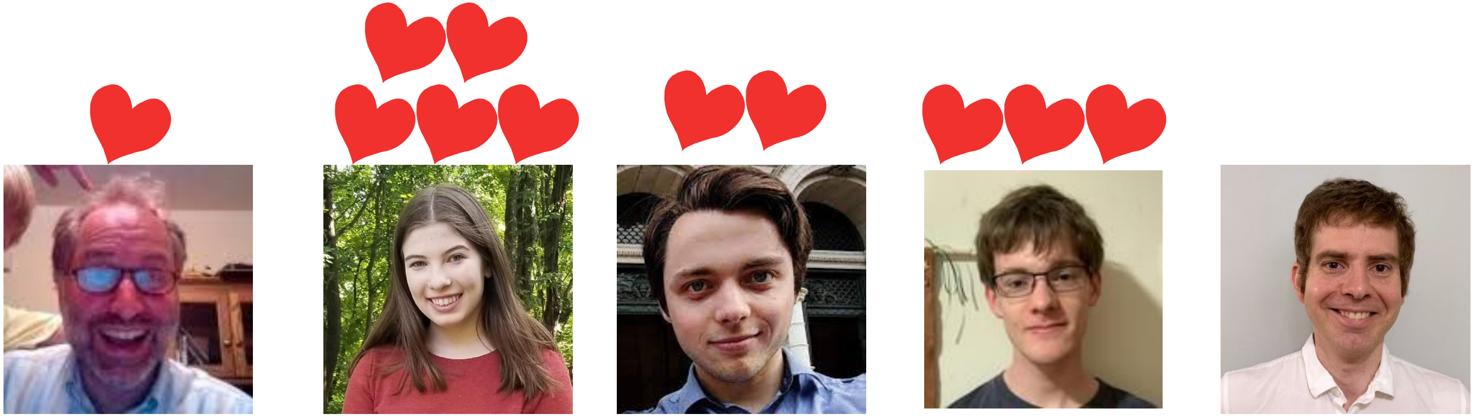
15451 Fall 2022

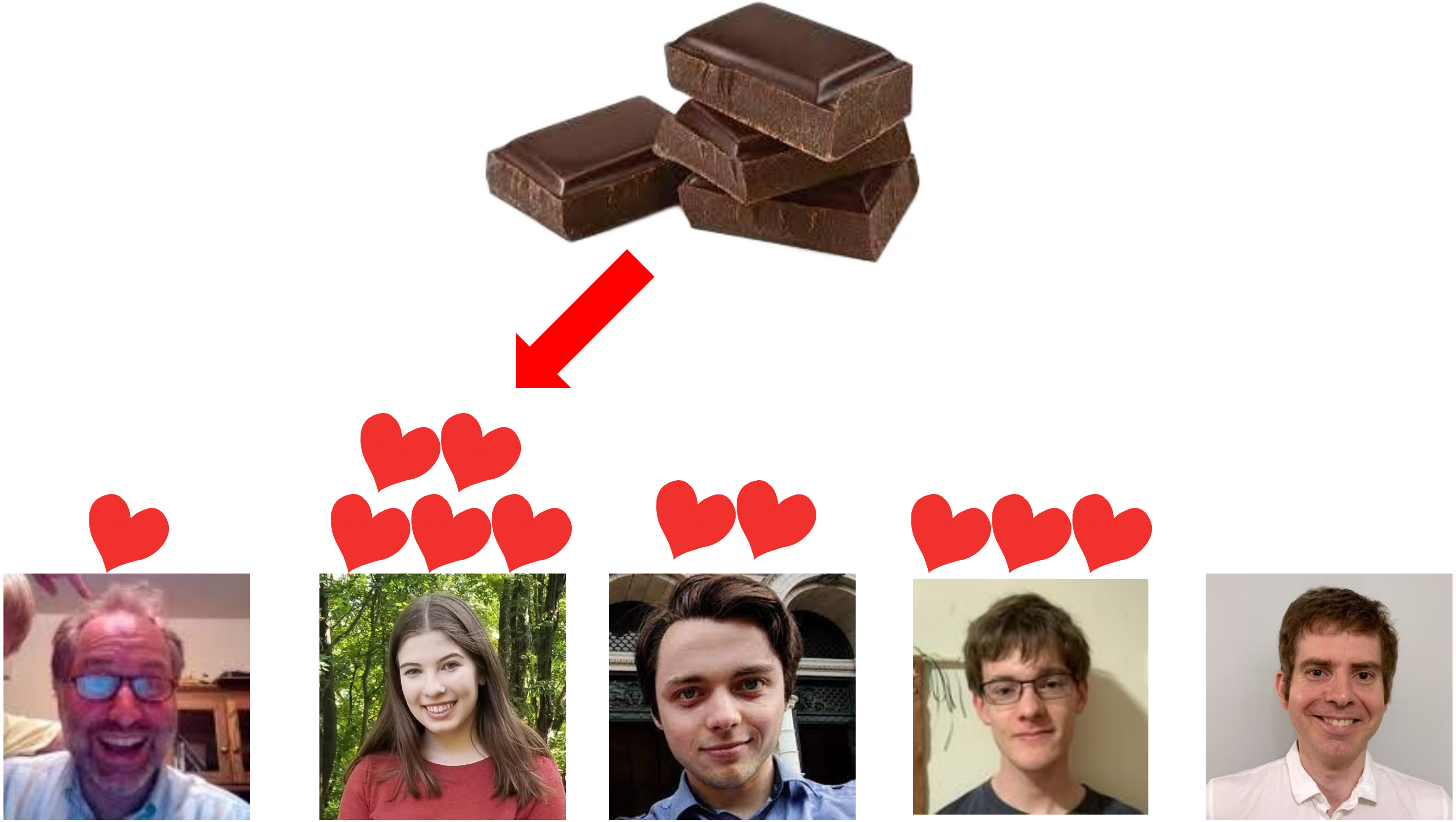
Mechanism Design

a.k.a **inverse game theory**

Elaine Shi

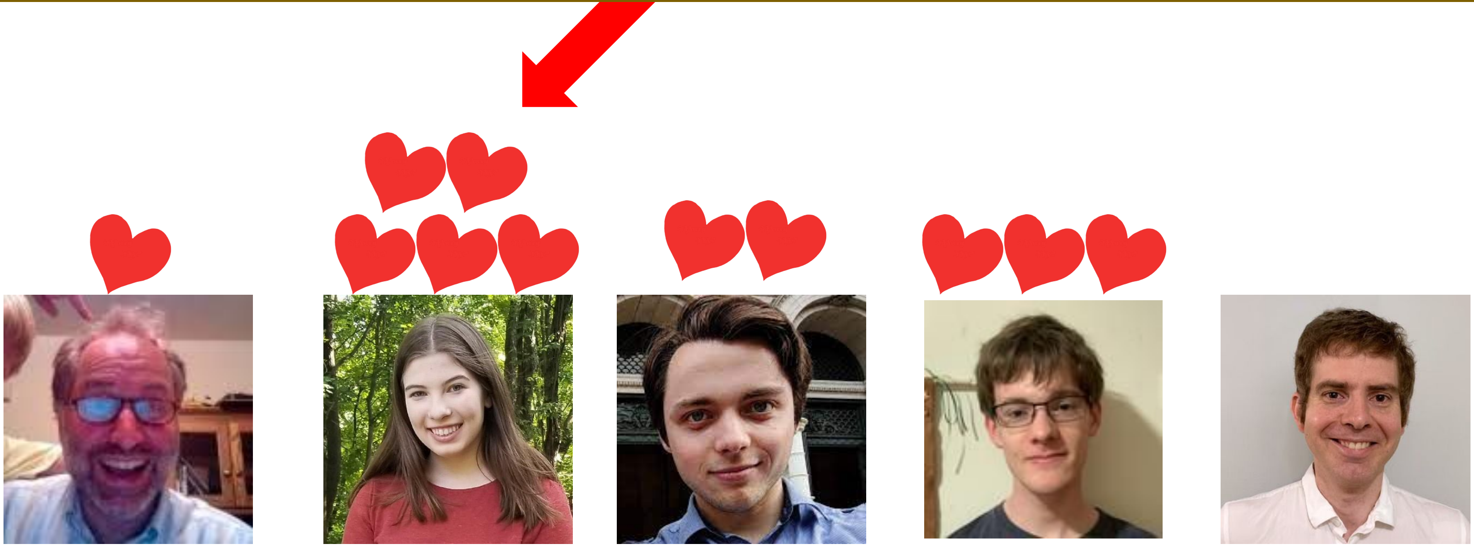
to give away





Maximizes **social welfare**

= sum of happiness



Let's do this in class

On a scale of **0-5** how much do you like

Problem:

Everyone is incentivized to report 5

i.e., mechanism incentivizes lying

How do I give away

such that

Maximize social welfare

Incentivize truthful bidding

Mechanism Design

a.k.a. inverse game theory

- Maximize social welfare
- Incentivize truthful bidding

Suppose we can charge **$$**

- Everyone bids
- Highest bidder wins
- Winner pays its bid

“1st price auction”

- Everyone bids
- Highest bidder wins
- Winner pays its bid

- v_i : true value of i-th bidder
- b_i : i-th bid
- **Allocation rule:** $\mathbf{x}(\mathbf{b})$ where $\mathbf{b} = (b_1 \ldots b_n)$
- **Payment rule:** $\mathbf{p}(\mathbf{b}) \rightarrow (p_1 \ldots p_n)$

p_i : payment of the ith bidder

- v_i : true value of i-th bidder
- b_i : i-th bid
- p_i : payment of bidder i
- **Utility** of bidder i:

$$u_i = \begin{cases} v_i - p_i & \text{if i wins} \\ 0 & \text{o.w.} \end{cases}$$

Example:

1st price auction: anyone who bids honestly has utility 0

Maximize social welfare:
give to whoever has the highest true value

Maximize social welfare:

give to whoever has the highest true value

i*'s utility

payment, auctioneer's utility

Social welfare = $v_{i^*} - p_{i^*} + p_{i^*} = v_{i^*}$

where i^* = winner

Does 1st price auction **incentivize truthful bidding?**

- Everyone bids
- Highest bidder wins
- Winner pays its bid

Does 1st price auction **incentivize truthful bidding?**

Incentivized to **underbid**

Example:

- My true value = 10
- I know that the 2nd highest bidder bids 9

What should I do?

2nd price auction a.k.a. Vickrey auction

- Highest bidder wins
- Winner pays **2nd highest bid**

winner → 10

payment → 9

10 9 8 5 1

Claim: 2nd price auction is **dominant-strategy incentive-compatible** , i.e.,

"DSIC"

Claim: 2nd price auction is **dominant-strategy incentive-compatible**, i.e.,

for any valuations $v_1, v_2, \ldots, v_n$, for any player i, for any bid vector $\mathbf{b}_{-i}$ of all other players, and any v_i',

bid vector for all users except i

$$u_i(\text{Vickrey}(v_i, \mathbf{b}_{-i})) \geq u_i(\text{Vickrey}(v_i', \mathbf{b}_{-i}))$$

i's utility if it bids truthfully

i's util when it misreports

where $(v_i', \mathbf{b}_{-i})$ means letting everyone else's bids be $\mathbf{b}_{-i}$ and let the i-th bid be v_i'

Even after seeing others' bid, I still want to bid truthfully

for any valuations $v_1, v_2, \ldots, v_n$, for any player i,
for any bid vector $\mathbf{b}_{-i}$ of all other players, and any v_i',

$$u_i(\text{Vickrey}(v_i, \mathbf{b}_{-i})) \geq u_i(\text{Vickrey}(v_i', \mathbf{b}_{-i}))$$

where $(v_i', \mathbf{b}_{-i})$ means letting everyone else's bids be $\mathbf{b}_{-i}$ and let the i-th bid be v_i'

Even after seeing others' bid, I still want to bid truthfully

also called an ex-post Nash equilibrium

for any valuations $v_1, v_2, \ldots, v_n$, for any player i,
for any bid vector $\mathbf{b}_{-i}$ of all other players, and any v_i',

$$u_i(\text{Vickrey}(v_i, \mathbf{b}_{-i})) \geq u_i(\text{Vickrey}(v_i', \mathbf{b}_{-i}))$$

where $(v_i', \mathbf{b}_{-i})$ means letting everyone else's bids be $\mathbf{b}_{-i}$ and let the i-th bid be v_i'

Claim: 2nd price auction is **dominant-strategy incentive-compatible**

Proof: Fix an arbitrary player i, and let b^* be the highest bid among b_{-i}

Case 1: $v_i > b^*$: if i bids truthfully, it util is $v_i - b^*$
if i bids anything above b^*, its util is still $v_i - b^*$
if i bids b^*, then its util is either $v_i - b^*$ or 0
if i bids $< b^*$ then its util is 0

Case 2: $v_i = b^*$: its util $= 0$, it doesn't care

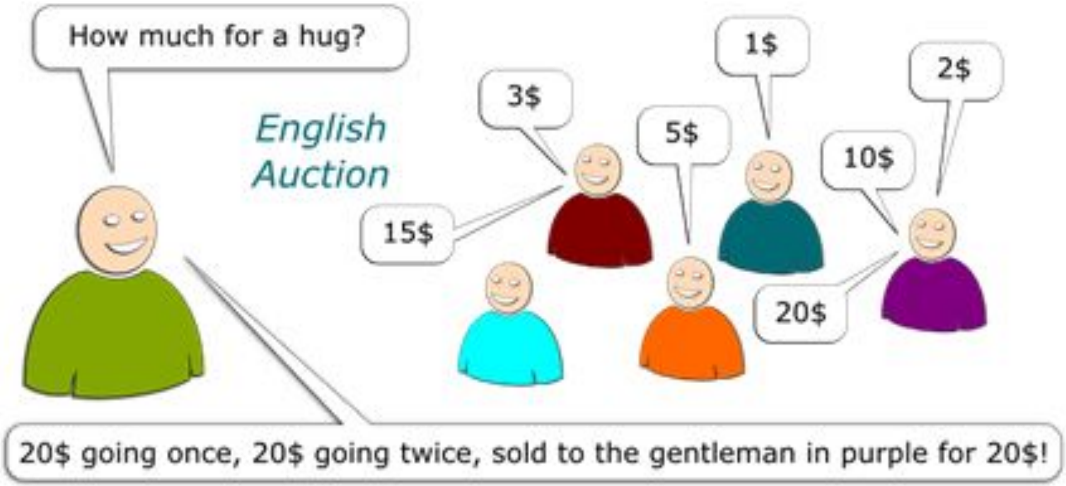
Case 3: $v_i < b^*$: if i bids truthfully, util $= 0$
if i bids $< b^*$, util $= 0$, if i bids $\geq b^*$, util ≤ 0

2nd price auction in the real world

-10

$-1+\epsilon$

-1



Multiple identical items, say 2 items

Multiple identical items, say 2 items

Top 2 bidders are winners.
Top bidder pays 2nd price.
2nd top bidder pays 3rd price.

Is this dominant-strategy incentive compatible?

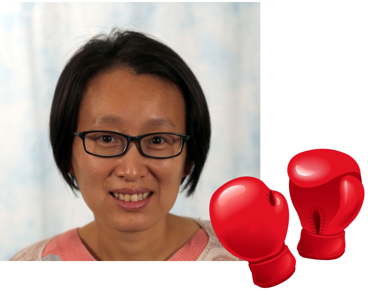
Second try

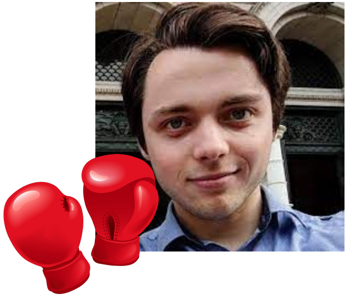
Top 2 bidders are winners.
Both winners pay 3rd price.

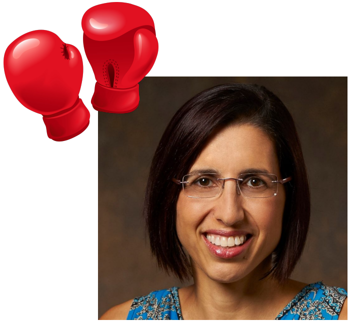
Is this dominant-strategy incentive compatible?

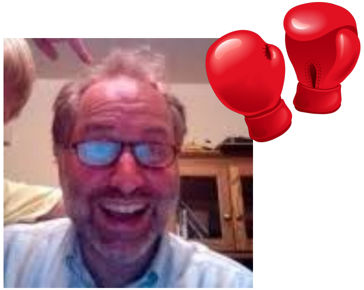
Vickrey-Clarke-Groves (VCG) Auction

Vickrey-Clarke-Groves (VCG) Auction

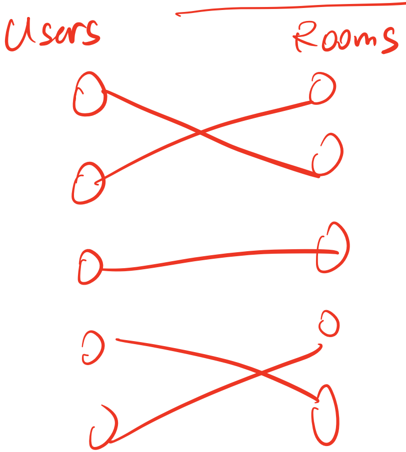








A: set of alternatives (or "allocations")



A: set of all possible perfect matchings

A: set of alternatives (or "allocations")

$v_i : A \to \mathbb{R}_{\geq 0}$: maps allocations to valuations

A: set of alternatives (or "allocations")

$v_i : A \to \mathbb{R}_{\geq 0}$: maps allocations to valuations

allocation payment

$u_i(a, p)$: utility of user i, $u_i(a, p) = v_i(a) - p$

$SW(a)$: social welfare $SW(a) = \sum_i v_i(a)$

Example

Direct revelation mechanism

- Everyone reveals their true values
  $\mathbf{v} = (v_{1,}\ v_{2,} \dots\ v_n)$
- **Allocation rule:** $f(\mathbf{v}) = a$ ∈ A
- **Payment rule:** $\mathbf{p}(\mathbf{v}) = (p_1, \dots\ p_n)$

A direct revelation (f, p) mechanism is incentive compatible iff

for every $\mathbf{v} = (v_1, \ldots, v_n)$, every i, every v_i', we have

$$v_i(f(\mathbf{v})) - p_i(\mathbf{v}) \geq v_i(f(v_i', \mathbf{v}_{-i})) - p_i(v_i', \mathbf{v}_{-i})$$

util of player i when it reports truthfully

util of player i when it misreports

A direct revelation (f, p) mechanism is incentive compatible iff

for every $\mathbf{v} = (v_1, \ldots, v_n)$, every i, every v_i', we have

$$v_i(f(\mathbf{v})) - p_i(\mathbf{v}) \geq v_i(f(v_i', \mathbf{v}_{-i})) - p_i(v_i', \mathbf{v}_{-i})$$

Misreporting never helps

<u>Claim:</u> **There is a mechanism that**

- Is dominant-strategy incentive compatible
- Maximizes social welfare if everyone bids truthfully

Claim: **There is a mechanism that**

- Is dominant-strategy incentive compatible
- Maximizes social welfare if everyone bids truthfully

"The VCG auction"

1st try

- $f(\mathbf{v})$ = the allocation that maximizes social welfare w.r.t **v**
- $p_i(\mathbf{v})$ = sum of everyone else's reported valuations

i.e., $p_i(\mathbf{v}) = -\sum_{j \neq i} v_j(f(\mathbf{v}))$

Analysis

If player i reports truthfully, then

$$u_i = v_i(f(\mathbf{v})) + \sum_{j \neq i} v_j(f(\mathbf{v})) = \sum_j v_j(f(\mathbf{v})) = \max_a \sum_j v_j(a)$$

Analysis

If player i reports truthfully, then

$$u_i = v_i(f(\mathbf{v})) + \sum_{j \neq i} v_j(f(\mathbf{v})) = \sum_j v_j(f(\mathbf{v})) = \max_a \sum_j v_j(a)$$

What if player i misreports v'_i ?

Analysis

If player i reports truthfully, then

$$u_i = v_i(f(\mathbf{v})) + \sum_{j \neq i} v_j(f(\mathbf{v})) = \sum_j v_j(f(\mathbf{v})) = \max_a \sum_j v_j(a)$$

If player i misreports v'$_i$, then

(v_i', v_{-i})

$$u_i = v_i(f(\mathbf{v}')) + \sum_{j \neq i} v_j(f(\mathbf{v}')) = \sum_j v_j(f(\mathbf{v}')) \leq \max_a \sum_j v_j(a)$$

Misreporting does not help

If player i reports truthfully, then

$$u_i = v_i(f(\mathbf{v})) + \sum_{j \neq i} v_j(f(\mathbf{v})) = \sum_j v_j(f(\mathbf{v})) = \max_a \sum_j v_j(a)$$

If player i misreports v'_i, then

$$u_i = v_i(f(\mathbf{v}')) + \sum_{j \neq i} v_j(f(\mathbf{v}')) = \sum_j v_j(f(\mathbf{v}')) \leq \max_a \sum_j v_j(a)$$

Problem: auctioneer to give money to bidders

Add to each $p_i(v)$ something that depends only on $\mathbf{v}_{-i}$

VCG -- general version

- f(**v**) = the allocation that maximizes social welfare w.r.t **v**
- $p_i(\mathbf{v}) = h_i(\mathbf{v}_{-i}) - \sum_{j \neq i} v_j(f(\mathbf{v}))$

How to choose $h_i(\mathbf{v}_{-i})$

- Auctioneer does not pay
- Every bidder has non-negative utility

$$h_i(\mathbf{v}_{-i}) = \max_a \sum_{j \neq i} v_j(a)$$

VCG -- standard version

- f(**v**) = the allocation that maximizes social welfare w.r.t **v**
- $p_i(\mathbf{v})$ = $\max_a \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(f(\mathbf{v}))$

VCG -- standard version

- $f(\mathbf{v})$ = the allocation that maximizes social welfare w.r.t $\mathbf{v}$
- $p_i(\mathbf{v})$ = $\max_a \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(f(\mathbf{v}))$

$$= v_i - \left(\sum_j v_j(f(\mathbf{v})) - \max_a \sum_{j \neq i} v_j(a) \right)$$

Example

2nd price auction is a special case of VCG auction