

# Lecture 23: Graph Compression

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# Outline

- Motivating Questions

- Spanners

- Multiplicative

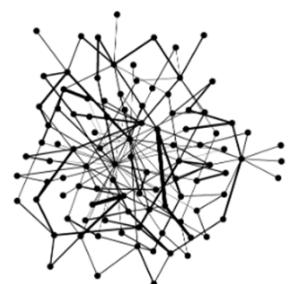
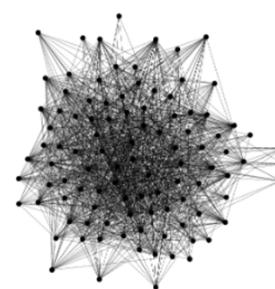
- Additive

# Diameter

- You have an unweighted, undirected graph  $G = (V, E)$  on  $n$  vertices
- Diameter is  $D_G = \max_{u, v \text{ in } V} d_G(u, v)$ , where  $d_G(u, v)$  is shortest path distance
- *How to compute  $D_G$ ?*
  - All pairs shortest path algorithm such as Floyd-Warshall
    - Time is  $O(|E| n)$  which could be  $O(n^3)$
    - Can we compute or approximate  $D_G$  faster?

# Shortest Path Queries

- $G = (V, E)$  is an unweighted, undirected graph on  $n$  vertices
- Want to answer shortest path queries: what is  $d_G(u, v)$ ?
- $|E|$  can be  $\Theta(n^2)$ , so want to “compress”  $G$  to fit in memory, but still want to answer shortest path queries
- Replace  $G$  with a subgraph  $H = (V, E')$ 
  - Store  $H$  instead of  $G$
  - Given query  $d_G(u, v)$ , respond with  $d_H(u, v)$
- Suppose  $G = (V, E)$  is a clique
  - *If  $\{u, v\}$  not in  $H$ , what is  $d_G(u, v)$  and what is  $d_H(u, v)$ ?*
- Can we find a small subgraph  $H$  to approximate  $d_G(u, v)$  for all  $u, v$ ?



# Outline

- Motivating Questions

- **Spanners**

- Multiplicative
  - Additive

# Spanners

- $G = (V, E)$  is undirected, unweighted graph on  $n$  vertices
- $d_G(u, v)$  is shortest path distance from  $u$  to  $v$
- A  $(k, b)$ -spanner of  $G$  is a subgraph  $H = (V, E')$  such that for all  $u, v$  in  $V$   
$$d_G(u, v) \leq d_H(u, v) \leq k \cdot d_G(u, v) + b$$
- If  $b = 0$ ,  $H$  is a **multiplicative** spanner
- If  $k = 1$ ,  $H$  is an **additive** spanner
- Do there exist  $(k, b)$ -spanners  $H$  with small  $|E'|$ ?

# Application of Spanners

- Shortest path query  $d_G(u, v)$ 
  - Replace  $G$  with a  $(k, b)$ -spanner  $H$  with  $|E'|$  edges
  - Output  $d_H(u, v)$
  - **Approximation:**  $d_G(u, v) \leq d_H(u, v) \leq k \cdot d_G(u, v) + b$
  - **Space:**  $O(|E'| + n)$  instead of  $O(|E| + n)$
  - **Time:**  $O(|E'| + n)$  instead of  $O(|E| + n)$
- Approximating diameter  $D_G$ 
  - Replace  $G$  with a  $(k, b)$ -spanner  $H$  with  $|E'|$  edges
  - Output diameter  $D_H$  of  $H$
  - **Approximation:**  $D_G \leq D_H \leq k \cdot D_G + b$
  - **Time:**  $O(n|E'|)$  instead of  $O(n|E|)$
- Faster if  $|E'| \ll |E|$ , but have to account for the time to create  $H$

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# Multiplicative Spanners

- A  $(k, b)$ -spanner of  $G$  is a subgraph  $H = (V, E')$  such that for all  $u, v$  in  $V$   
$$d_G(u, v) \leq d_H(u, v) \leq k \cdot d_G(u, v) + b$$
- If  $b = 0$ , then  $d_G(u, v) \leq d_H(u, v) \leq k \cdot d_G(u, v)$  for all  $u, v$  in  $V$ 
  - $H = (V, E')$  is a  $k$ -multiplicative spanner
  - *How small can  $|E'|$  be?*
- If  $d_G(u, v) = 1$ , then  $d_H(u, v) \leq k$
- Conversely, if  $d_H(u, v) \leq k$  for all *edges*  $\{u, v\}$  of  $G$ , then for any vertices  $u', v' \in V$ ,  
$$d_H(u', v') \leq k \cdot d_G(u', v')$$
- To construct  $H$ , just need for all edges  $\{u, v\}$  in  $E$ ,  $d_H(u, v) \leq k$

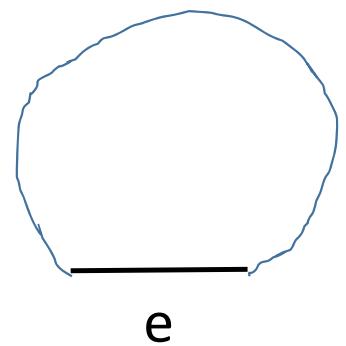
# Greedy Algorithm for Multiplicative Spanners

- Let's build  $H = (V, E')$  by walking through the edges of  $G$

- Initialize  $H = (V, \emptyset)$

- For each edge  $e$  in  $G$

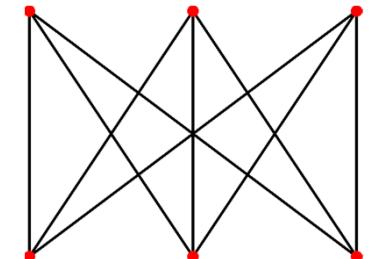
- If \_\_\_\_\_, then include  $e$  in  $H$



- *That's the algorithm! What should \_\_\_\_\_ be?*
  - “If  $e$  doesn't form a cycle of length at most  $k+1$  with the edges you've already included”
- *Why is this correct?*
  - For each edge not included, there's a path of length at most  $k$  between its endpoints
- How many edges does  $H$  have?

# Bounding the Number of Edges in $H$

- $H$  doesn't have a cycle of length at most  $k+1$ . *Why?*
- Minimum cycle length is called the girth
- What's the maximum number of edges in a graph with girth at least  $k+2$ ?
  - *What if  $k = 2$ ?*
  - A complete bipartite graph has  $\Omega(n^2)$  edges, and girth 4
  - *What if  $k = 3$ ?*
  - At most  $O(n^{\frac{3}{2}})$  edges!
  - For  $k=2t$  or  $k=2t-1$  for an integer  $t$ , at most  $O(n^{1+\frac{1}{t}})$  edges, so  $H$  is tiny!



# Bounding the Number of Edges in H

- **Theorem:** for  $k=2t$  or  $k=2t-1$ , a graph with girth at least  $k+2$  has  $O(n^{1+\frac{1}{t}})$  edges
- **Lemma:** let  $\bar{d} = 2m/n$  be the average degree in a graph  $G$  with  $m$  edges and  $n$  nodes. There is a non-empty subgraph  $G'$  of  $G$  with minimum degree  $\bar{d}/2$
- Proof: Initialize  $V_0 = V$  and  $E_0 = E$ 
  - $i = 0$
  - While there is a vertex  $v$  of degree at most  $|E_i|/|V_i|$ ,
    - $i \leftarrow i + 1$
    - $V_i \leftarrow V_{i-1} \setminus \{v\}$
    - $E_i \leftarrow E_{i-1} \setminus \{\{v, w\} \text{ for all neighbors } w \text{ of } v\}$
  - Output  $G' = (V_i, E_i)$
  - $G'$  is non-empty because  $\frac{|E_i|}{|V_i|} \geq \frac{|E_{i-1}|}{|V_{i-1}|} \geq \dots \geq \frac{|E|}{|V|} = \frac{m}{n} > 0$

For  $t \leq \frac{x}{y}$  we have:

$$\frac{x-t}{y-1} \geq \frac{x - \frac{x}{y}}{y-1} = \frac{x\left(1 - \frac{1}{y}\right)}{y\left(1 - \frac{1}{y}\right)} = \frac{x}{y}$$

# Bounding the Number of Edges in H

- **Theorem:** for  $k = 2t$  or  $k=2t-1$ , a graph with girth at least  $k+2$  has  $O(n^{1+\frac{1}{t}})$  edges
- Proof:
  - By lemma, a graph  $G$  has a non-empty subgraph  $G'$  with min degree  $\bar{d}/2$
  - Grow a breadth-first-search (BFS) tree from a node  $v \in G'$
  - $G'$  has girth  $k+2$
  - At level  $t$  in the BFS tree, there are at least  $\left(\frac{\bar{d}}{2} - 1\right)^t$  *distinct* nodes
  - $\left(\frac{\bar{d}}{2} - 1\right)^t \leq n$ , so  $\left(\frac{m}{n} - 1\right)^t \leq n$ , and solving gives  $m \leq n + n^{1+\frac{1}{t}}$

# Can we do Better?

- **Girth conjecture:** for  $k = 2t$  or  $k=2t-1$ , there are graphs with girth  $k+2$  and  $\Omega(n^{1+\frac{1}{t}})$  edges
- Implies any  $k$ -multiplicative spanner has  $\Omega(n^{1+\frac{1}{t}})$  edges. *Why?*
- If we delete any edge  $\{u,v\}$  in  $G$ , the distance from  $u$  to  $v$  increases from 1 to  $k+1$
- Only  $k$ -spanner of  $G$  is  $G$  itself
- Girth conjecture true for  $k = 1, 2, 3, 5$

# Where are We?

- Can find a  $(2t-1)$ -spanner with  $O(n^{1+\frac{1}{t}})$  edges
- Can approximate  $d_G(u, v)$  for any  $u, v$  up to a multiplicative factor  $2k-1$
- Don't store  $G$ , just store  $H$ . Only  $O(|E'|) = O(n^{1+\frac{1}{t}})$  instead of  $O(n^2)$  edges
- Time to compute  $d_H(u, v)$ , given  $H$ , is  $O(|E'|) = O(n^{1+\frac{1}{t}})$ 
  - Faster than the  $O(n^2)$  time to query a dense graph  $G$
  - Greedy algorithm to find  $H$  is slow, but can find  $H$  in  $O(|E|+n)$  time

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# Additive Spanners

- A  $(k, b)$ -spanner of  $G$  is a subgraph  $H = (V, E')$  such that for all  $u, v$  in  $V$   
$$d_G(u, v) \leq d_H(u, v) \leq k \cdot d_G(u, v) + b$$
- If  $k = 1$ , then  $d_G(u, v) \leq d_H(u, v) \leq d_G(u, v) + b$  for all  $u, v$  in  $V$ 
  - $H = (V, E')$  is a  $b$ -additive spanner
  - *How small can  $|E'|$  be?*
- For multiplicative spanners, sufficient to show for all edges  $\{u, v\}$  in  $G$ ,  $d_H(u, v) \leq k$ 
  - Insufficient for additive spanners to show  $d_H(u, v) \leq b + 1$  for all edges  $\{u, v\}$  in  $G$
- **Would you believe:** there is a 2-additive spanner with  $O(n^{3/2} \log n)$  edges?

# Additive Spanner Algorithm

- The algorithm has two parts

(1) Include in  $H$  all edges incident to vertices of degree at most  $\sqrt{n}$

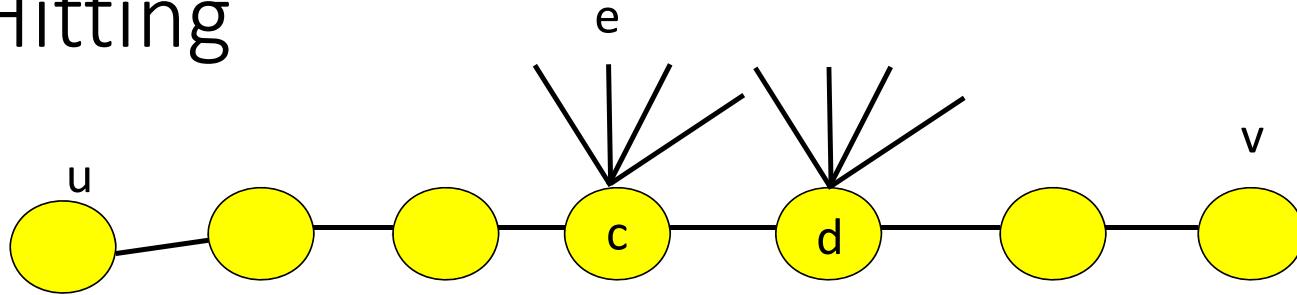
- at most  $n^{3/2}$  edges (**why?**)

(2) Randomly sample a set  $S$  of  $2\sqrt{n} \cdot \ln n$  vertices and include a BFS tree rooted at each vertex in  $S$ , in  $H$

- at most  $2n^{3/2} \ln n$  edges (**why?**)

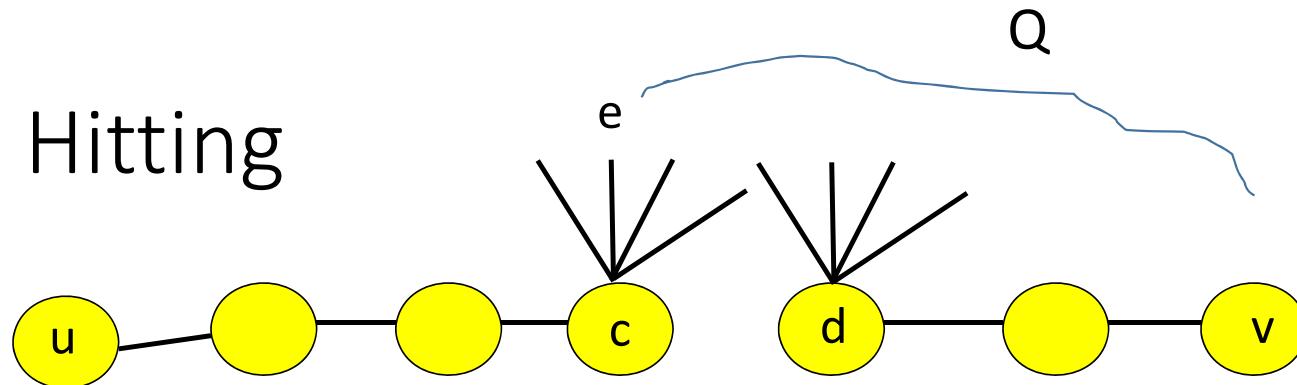
$H$  has  $O(n^{\frac{3}{2}} \log n)$  edges. Why is it a 2-additive spanner?

# Path Hitting



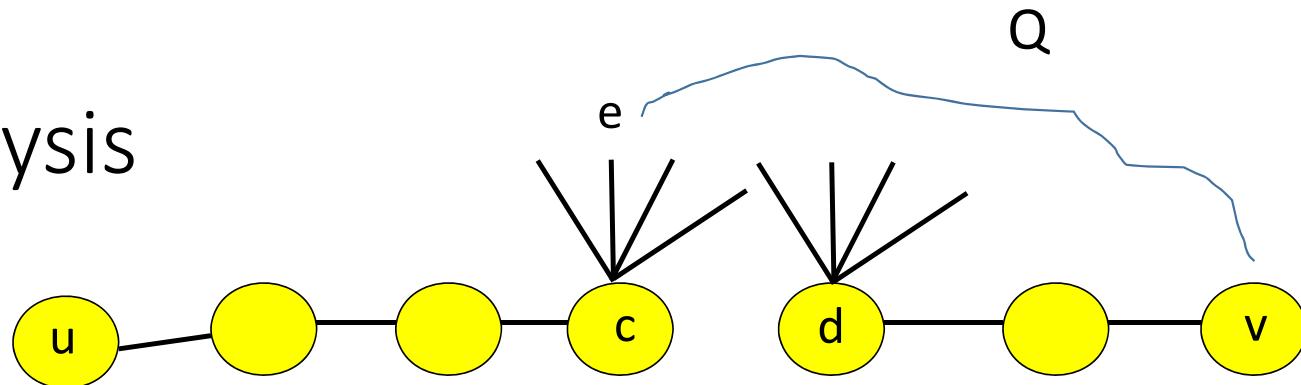
- Consider a shortest path  $P$  from  $u$  to  $v$  in  $G$
- If all nodes on  $P$  have degree  $\leq \sqrt{n}$ , then all edges in  $P$  are included in the spanner  $H$
- Otherwise consider the first edge  $\{c, d\}$  in  $P$ , but not in  $H$ 
  - $c$  and  $d$  have degree at least  $\sqrt{n}$
- Since we randomly sample a set  $S$  of size  $2\sqrt{n} \cdot \ln n$ , with high probability, we sample a neighbor  $e$  of  $c$  (probability we don't sample a neighbor of  $c$  at most  $\left(1 - \frac{\sqrt{n}}{n}\right)^{2\sqrt{n} \ln n} \leq \frac{1}{n^2}$ )

## Path Hitting



- For each of our sampled vertices in  $S$ , we grew a BFS tree
- Let  $T_e$  be the BFS tree rooted at  $e$  included in  $H$
- Let  $Q$  be the path from  $e$  to  $v$  in  $T_e$
- Consider the path  $P'$  in  $H$  which follows  $P$  from  $u$  to  $c$ , then traverses edge  $\{c, e\}$ , then follows  $Q$  to  $v$ . *How long is  $P'$ ?*

## Analysis



- Consider the path  $P'$  in  $H$  which follows  $P$  from  $u$  to  $c$ , then traverses edge  $\{c, e\}$ , then follows  $Q$  to  $v$ . **How long is  $P'$ ?**
- $Q$  is a shortest path from  $e$  to  $v$  in  $G$ !
- $d_Q(e, v) \leq 1 + d_P(c, v)$
- $d_{P'}(u, v) = d_P(u, c) + 1 + d_Q(e, v) \leq d_P(u, c) + d_P(c, v) + 2 = d_P(u, v) + 2$

# Additive Spanner Notes

- Can find a 2-additive spanner with  $O(n^{3/2} \log n)$  edges
  - Can get  $O(n^{3/2})$  edges
- Can find a 4-additive spanner with  $n^{7/5} \text{poly}(\log n)$  edges
- Can find a 6-additive spanner with  $O(n^{4/3})$  edges
- For any constant  $C > 0$ , any  $C$ -additive spanner requires  $\Omega(n^{4/3})$  edges