15-451/651 Algorithm Design & Analysis, Fall 2025

Recitation #9

Objectives

- Review and understand the concept of **LP duality**
- Practice taking duals of LPs and interpreting the results
- Understand how to take the dual of a nontrivial LP representing a graph problem
- See another example of integrality of LP solutions

Recitation Problems

1. **(Duality Recap)**

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{s.t.} \quad & Ax \le b \\ & x \ge 0 \end{aligned}$$

its dual is:

$$\begin{aligned} \text{minimize} \quad & b^T y \\ \text{s.t.} \quad & A^T y \ge c \\ & y \ge 0 \end{aligned}$$

Using this, given:

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 6x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 \le 5 \\ & 6x_1 + 3x_2 + 3x_3 \le 45 \\ & 2x_1 + x_2 + x_3 \le 3 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

what is the dual?

2. **(Duality in graph problems)** Recall that a *vertex cover* of a graph $G = (V, E)$ is a subset of the vertices such that every edge in E is adjacent to at least one of the vertices in the subset. The minimum vertex cover is a vertex cover with the fewest possible vertices.

   (a) Suppose we want to write a linear program to output the minimum vertex cover. Recall that linear programs are not guaranteed to output integer solutions. This makes outputting a vertex cover difficult for a linear program, since the natural choice would be to use indicator variables, where $x_v = 1$ if v is in the vertex cover, or $x_v = 0$ if v is not in the vertex cover (indicator variables are not valid in an LP!).

However, we can still write an *LP relaxation*, which is the integer program that we would like to write, but removes the integrality constraints. For the vertex cover problem, an LP relaxation means that vertices may be "partially" included in the vertex cover (this is often called the "fractional vertex cover" problem). Despite not modeling the problem exactly, LP relaxations often help us find bounds on the actual problem, and have other uses later in the course (approximation algorithms). Also, we can sometimes prove that the relaxation always has integral vertices and hence does admit integer optimal solutions!

Write an LP relaxation of the vertex cover problem.

(b) Write down the dual of this LP. What does it mean?

(c) Based on your answer to Part (a), which of the following are true, and why?

- ☐ For any graph G, size of minimum vertex cover = size of maximum matching
- ☐ For any graph G, size of minimum vertex cover ≤ size of maximum matching
- ☐ For any graph G, size of minimum vertex cover ≥ size of maximum matching

3. **(Integrality of flows)** In lecture we proved that given a graph G, the *matching polytope* MP_G defined by the constraints of the maximum matching problem always has integral vertices if G is bipartite.

Let's prove something more general. Recall that bipartite matching is really just a special case of maximum flow, which is a special case of minimum-cost flows.

Prove that for any given flow network G, the *flow polytope* FP_G defined by the LP constraints of the maximum flow / minimum-cost flow problem has integral vertices.

Variables: $f_{u,v}$ for each edge (u,v)

Objective: Maximize $\sum_{u\in V} f_{s,u} - \sum_{v\in V} f_{v,s}$

Constraints:

$$0 \le f_{u,v} \le c(u,v) \forall (u,v) \in E \quad \text{(capacity)}$$

$$\sum_{u\in V} f_{u,v} = \sum_{w\in V} f_{v,w} \forall v \in V \setminus \{s,t\} \quad \text{(flow conservation)}$$

(a) Consider a feasible solution $f = \{f_e\}$ that contains at least one fractional value. To make the problem easier, here are two cases for the structure that the fractional values might appear in.

- **Case 1:** There exists an $s-t$ path of edges (*ignoring their direction*) with non-integral flows on every edge.
- **Case 2:** There exists a cycle of edges (*ignoring their direction*) with non-integral flows on every edge.

Explain why these cases are exhaustive, i.e., why if a feasible solution is fractional, it must correspond to at least one of these cases.

(b) Prove that in both of these cases, such a fractional feasible solution f does **not** correspond to a vertex solution of the LP.