# 15-451/651 Algorithm Design & Analysis, Fall 2025

# Recitation #8

**Objectives**

- Understand the concept of a two-player zero-sum game
- Practice using the guess and bound technique to solve a two-player zero sum game
- Practice writing linear programs
- Practice solving small zero-sum games by graphing and optimizing and understanding dominant strategies

## Recitation Problems

1. **(A matrix game)** Consider an $n \times n$ matrix $A = \{a_{i,j}\}$ in which the elements on the diagonal are positive, and all the rest are zero.

$$\begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}$$

Consider the two-player zero-sum game where the matrix $A$ is the payoff for the row player, i.e., the row player picks a row and the column player picks a column, and the payoff to the row player is the value in that entry of the matrix. Express the value of the game $V$ for the row player as a function of the non-zero values in the matrix.

**Hint:** You should use the *guess and bound* strategy, i.e., intuit a strategy for the row player which you think might be optimal, then show a *lower bound for that strategy* (show that the strategy achieves a value of at least $V$ for the row player), and then intuit a strategy for the column player which seems optimal and show an *upper bound for that strategy* (it prevents the row player from obtaining a value larger than $V$).

   (a) As a starting point, suppose we try the strategy of uniformly randomly picking a row, i.e., pick each row with probability $1/n$. What would the optimal column response be? What does this give as a lower bound for the value of the game?

   (b) Provide a better strategy for the row player and prove that it achieves a lower bound (to the row player) of $V$, where $V$ is a function of the non-zero entries of $A$.

   (c) Provide a corresponding strategy for the column player and prove that it achieves an upper bound (to the row player) of $V$, where $V$ is the same value as in Part (a). This proves that the value of the game to the row player is $V$.

2. **(Shortest paths as an LP)** You can formulate the $s$-$t$ *shortest-path problem* as a linear program. Actually there are multiple ways of doing it, so lets look at two of them! The input is a directed graph $G$ with edge weights $w(e) \geq 0$, start node $s$, and a target $t$. We want to find a path from $s$ to $t$ of least weight.

   **First method: Vertex variables.** Suppose we have a variable $d_v$ for every vertex $v$ representing its distance from $s$.

   (a) What are the constraints you should write in terms of the $d$ variables?

   Hint: $d_v$ represents the distance of the **shortest path** from $s$ to $v$.

   (b) What is the objective function in terms of the $d$ variables?

   **Second method: Edge variables.** Suppose we have a variable $x_e$ for every edge $e$, where $0 \leq x_e \leq 1$. We want these variables to represent which edges are on the shortest path and which are not, i.e., we want the variables to essentially be **indicator variables**.

   (c) What is a suitable objective function for the problem given these variables?

   (d) For the solution to make sense, we must select a set of edges that form an $s$-$t$ path. We cannot just pick arbitrary edges that are not touching each other. The tricky part is turning that requirement into a constraint. As a hint, this is almost identical to a *flow*. A path is just a set of edges where each vertex on the path has exactly one edge coming in and one edge coming out (just like flow conservation!). Given this hint, try to write down some constraints that work.

   (e) Do you have to make any assumptions about the nature of the solution for this to work? You are not required to prove these, but you should clearly state what you need to assume. Is this assumption valid in general?

3. **(Extra Zero Sum Games & Dominance Practice - Optional Problem)** Consider the following zero-sum game:

| | | **column player** | |
|---|---|---|---|
| | | **A** | **B** |
| **row player** | **1** | $(0, 0)$ | $(1, -1)$ |
| | **2** | $(3, -3)$ | $(2, -2)$ |

(a) Let $\mathbf{p} = (p, 1-p)$ be the row player's mixed strategy. As a function of $p$, what is the expected value to the row player $V_R(\mathbf{p}, \mathbf{q})$ when the column player plays column A or column B? Write both expressions, and then plot them against $p$.

(b) Based on your answer to part (a), what is the minimax optimal strategy for the row player, and what value is the value of the game?

It turns out that for this particular game, there's an even faster way to find this optimal strategy by observing that in every column, row 2 gives a better payoff for the row player than row 1. When this occurs, we say that row 2 dominates row 1.

In general we can say that pure strategy $\alpha$ dominates pure strategy $\beta$ when

| | **row strategies** | **column strategies** |
|---|---|---|
| **strictly dominates** | $R_{\alpha j} > R_{\beta j} \quad \forall j$ | $C_{i\alpha} > C_{i\beta} \quad \forall i$ |
| **weakly dominates** | $R_{\alpha j} \geq R_{\beta j} \quad \forall j$ | $C_{i\alpha} \geq C_{i\beta} \quad \forall i$ |

and if a strategy is dominated by another strategy, we can say there is no reason to ever play it instead of that strategy. This can enable us to solve games faster, or even solve larger games than we would otherwise be able to. Consider for example, the following game:

| | | **column player** | | |
|---|---|---|---|---|
| | | **A** | **B** | **C** |
| **row player** | **1** | $(1, -1)$ | $(2, -2)$ | $(4, -4)$ |
| | **2** | $(2, -2)$ | $(-1, 1)$ | $(0, 0)$ |
| | **3** | $(2, -2)$ | $(3, -3)$ | $(1, -1)$ |

(c) Find and eliminate any dominated strategies. What game matrix does this leave you with?

(d) Now solve the resulting matrix to find the minimax optimal strategy for the row and column players and the value of the game.