**15-451/651 Algorithm Design & Analysis**

**Fall 2022, Recitation #8**

**Objectives**

- To review zero-sum games and how to determine their values and optimal strategies.
- To practice constructing the dual of a linear program.
- To explore the primal and dual relationship in classic graph problems.

# Recitation Problems

1. **(Practice with Zero-sum Games)**

   (a) Consider a zero-sum game with payoffs:

| | | **A** | **B** |
|---|---|---|---|
| **row** | **1** | $(2, -2)$ | $(-1, 1)$ |
| **player** | **2** | $(-3, 3)$ | $(4, -4)$ |

   Find the minimax optimal strategies for both players and the value of the game.

(b) Now consider the game with payoffs:

| | | column player | |
|---|---|---|---|
| | | **A** | **B** |
| **row** | **1** | $\left(-\frac{1}{2}, \frac{1}{2}\right)$ | $(-1, 1)$ |
| **player** | **2** | $(1, -1)$ | $\left(\frac{2}{3}, -\frac{2}{3}\right)$ |

Find the minimax optimal strategies for both players and the value of the game.

2. **(Duality Practice)** Given a primal LP

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

its dual is:

$$\begin{aligned} \text{minimize} \quad & y^T b \\ \text{s.t.} \quad & y^T A \geq c^T \\ & y \geq 0 \end{aligned}$$

For example, given

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 6x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 && \leq 5 \\ & 6x_1 + 3x_2 + 3x_3 && \leq 45 \\ & 2x_1 + x_2 + x_3 && \leq 3 \\ & x_1, x_2, x_3 && \geq 0 \end{aligned}$$

What is the dual?

3. **(Duality in graph problems)** Recall that a *vertex cover* of a graph $G = (V, E)$ is a subset of the vertices such that every edge in $E$ is adjacent to at least one of the vertices in the subset. The minimum vertex cover is a vertex cover with the fewest possible vertices.

   (a) Write down a linear program for the vertex cover problem. You might need to make an assumption. What is not quite exact about this LP?

   (b) Write down the dual of this LP. What does it mean?

   (c) Based on your answer to Part (a), which of the following are true, and why?

   - ☐ For any graph $G$, size of minimum vertex cover = size of maximum matching
   - ☐ For any graph $G$, size of minimum vertex cover $\leq$ size of maximum matching
   - ☐ For any graph $G$, size of minimum vertex cover $\geq$ size of maximum matching

4. **(Seidel's situations)** Consider the *deterministic* version of Seidel's 2D linear programming algorithm.

   (a) Describe a family of constraints that will lead to the *best-case* behaviour for the algorithm. Your family should not contain any redundant (parallel) constraints.

   (b) Describe a family of constraints that will lead to the *worst-case* behaviour of the algorithm. Your family should not contain any redundant (parallel) constraints.

# Further Review

1. **(Short answer / multiple choice)**

   (a) Which of the following algorithms solve a linear program in polynomial time?

   ☐ The Simplex Algorithm

   ☐ The Ellipsoid Algorithm

   ☐ Karmarkar's Algorithm

   ☐ Seidel's Algorithm

   (b) What is the dual of this linear program?

$$\begin{aligned} \text{minimize} \quad & 2y_1 - 3y_2 \\ \text{s.t.} \quad & y_1 - 4y_2 \geq 6 \\ & 3y_1 + 2y_2 \leq 4 \\ & y_1, y_2 \geq 0 \end{aligned}$$

   (c) Convert this LP into standard form

$$\begin{aligned} \text{minimize} \quad & x_1 - 5x_2 + 3x_3 \\ \text{s.t.} \quad & 5x_1 + 4x_2 - x_3 \geq 6 \\ & 3x_1 + 2x_2 + x_3 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

   (d) Convert the following LP to standard form, and then write its dual.

$$\begin{aligned} \text{minimize} \quad & -x_1 + 2x_2 \\ & x_1 \leq 9 \\ & 3x_2 + x_1 \geq 14 \end{aligned}$$

   (e) Suppose we have a primal LP with an objective coefficient vector $c$ and it's dual with objective coefficient vector $b$. We then find an assignment $x$ that satisfies the primal and an assignment $y$ that satisfies the dual. Select the strongest statement that must be true.

   ☐ $c^T x = b^T y$

   ☐ $c^T x \leq b^T y$

   ☐ $c^T x \geq b^T y$

2. **(Another 2-Row Game)** Suppose the row pay-off matrix is given as such:

| | $A$ | $B$ | $C$ | $D$ |
|---|---|---|---|---|
| 1 | 3 | 2 | 1 | 3 |
| 2 | 4 | 1 | 3 | 0 |

(a) What is the optimal strategy for the row player?

(b) What is the optimal strategy for the column player?

3. **(Rock-Paper-Scissors with a Twist)**

Suppose we have a non-standard game of rock-paper-scissors, which is still zero-sum, but with the following payoffs for the row player (Alice):

| | | Bob plays | | |
|---|---|---|---|---|
| | | $r$ | $p$ | $s$ |
| Alice plays | $r$ | 0 | $-1$ | 2 |
| | $p$ | 1 | 0.5 | $-1$ |
| | $s$ | $-1$ | 2 | $-1$ |

If Alice decides to play $\mathbf{p} = (p_1, p_2, 1 - p_1 - p_2)$ as her strategy, what should Bob play to minimize the payoff to Alice? What is Alice's payoff if he does this?

4. **(Shortest paths as an LP)** You can code up the $s$-$t$ shortest-path problem as an LP. Actually there are multiple ways of doing it, so lets look at two of them! The input is a directed graph $G$ with edge weights $w(e) \geq 0$, start node $s$, and a target $t$. We want to find a path from $s$ to $t$ of least weight.

**First method: Vertex variables.** Suppose we have a variable $d_v$ for every vertex $v$, representing its distance from $s$.

(a) What are the constraints you should write in terms of the $d$ variables?

(b) What is the objective function in terms of the $d$ variables? (Hint: counter-intuitively, you actually want to maximize, not minimize)

**Second method: Edge variables.** Suppose we have a variable $f_e$ for each edge $e$, where $0 \leq f_e \leq 1$. We want these variables to represent which edges are on the shortest path and which are not (imagine we want $f_e = 1$ if $e$ is on the shortest path, and $f_e = 0$ if not). Another way to say this is that we can think of $f_e$ *like a flow* on the edges $e$, and our goal is to send a unit of flow from $s$ to $t$ along the shortest path.

(c) Keeping in mind that the $f_e$ variables should behave like a flow, write down some suitable constraints.

(d) What is a suitable objective function for the problem?

(e) Do you have to make any assumptions about the nature of the solution for this to work? You are not required to prove these, but you should clearly state what you need to assume.

**Duality.** Lastly, lets see how duality comes into play.

(f) Take the dual of the first linear program (the one with vertex variables). What do you get?

5. **(Maximum-flow, minimum-what?)** Let $G = (V, E)$ be a directed graph with edge capacities $c(u, v)$ for $(u, v) \in E$. Recall from lecture that the max-flow problem can be

written as an LP. Defining a flow variable $f_{uv}$ representing $f(u,v)$ for every $(u,v) \in E$, we have the LP

$$\begin{aligned}
\text{maximize} \quad & \sum_{(s,u)\in E} f_{su} - \sum_{(u,s)\in E} f_{us}. && \text{(the net } s\text{-}t \text{ flow)} \\
\text{s.t.} \quad & \sum_{\substack{v \text{ s.t.} \\ (v,u)\in E}} f_{vu} - \sum_{\substack{v \text{ s.t.} \\ (u,v)\in E}} f_{uv} = 0. && \text{for all } u \notin \{s,t\} \text{ (flow conservation)} \\
& 0 \le f_{uv} \le c(u,v) && \text{for all } (u,v) \in E \text{ (capacity constraints)}
\end{aligned}$$

In standard form, this becomes

$$\begin{aligned}
\text{maximize} \quad & \sum_{(s,u)\in E} f_{su} - \sum_{(u,s)\in E} f_{us}. && \text{(the net } s\text{-}t \text{ flow)} \\
\text{s.t.} \quad & \sum_{\substack{v \text{ s.t.} \\ (v,u)\in E}} f_{vu} - \sum_{\substack{v \text{ s.t.} \\ (u,v)\in E}} f_{uv} \le 0 && \text{for all } u \notin \{s,t\}, \text{ (flow conservation)} \\
& \sum_{\substack{v \text{ s.t.} \\ (u,v)\in E}} f_{uv} - \sum_{\substack{v \text{ s.t.} \\ (v,u)\in E}} f_{vu} \le 0 && \text{for all } u \notin \{s,t\}, \text{ (flow conservation)} \\
& f_{uv} \le c(u,v) && \text{for all } (u,v) \in E, \text{ (capacity constraints)} \\
& f_{uv} \ge 0 && \text{for all } (u,v) \in E, \text{ (capacity constraints)}
\end{aligned}$$

Now that this LP is in standard form, we can apply our usual rules to take the dual.

(a) What is the dual of this problem?

(b) Simplify the dual LP as much as possible, which will involve making it *not* standard form. **Hints:**

- Remember that when we learned how to convert arbitrary LPs into standard form, we made substitutions like switching unbounded variables $x$ with two non-negative variables $x^+ - x^-$. You might be able to simplify the resulting LP by doing the opposite.

- $s$ and $t$ are special cases (they don't have conservation constraints) which means they will also end up as special cases in the dual. Creating some extra variables for them will help to eliminate redundant constraints.

(c) Intuit that this corresponds to min-cut. If you assume you get an integer solution to this LP, describe what the variables and their values represent, and how each constraint forces the solution to be a minimum cut. It is possible to prove that you actually always will get an integer solution, but its a hard proof, so we won't do it.