1 ZERO-SUM GAMES

Consider a zero-sum game with payoffs:

		A	**B**
row	**1**	$(2, -2)$	$(-1, 1)$
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Find the minimax optimal strategies for both players and compute the value of the game.

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IF
row player chooses strategy 1
col player chooses strategy 0
THEN
row get $-1/2$, col gets $1/2$

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THM: PAYOFF

if the row player uses a mixed strategy $\vec{p}$ and the col player uses strategy $\vec{q}$...

the EXPECTED PAYOFF of the game for row is...

$$V_R(p, q) = \sum_{ij} p_i q_j R_{ij}$$

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THM: LOWER BOUND

the worst expected payoff for the row player is...

$$lb = \max_p \left(\min_q \left(v_R(p, q) \right) \right)$$

- wtf is going on here...?

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- suppose i am row and i picked some strategy p. p is fixed

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$$lb = \max_p \Big(\underbrace{\min_q \big(V_R(p,q) \big)} \Big)$$

- p is fixed. $\min(V_R(p,q))$ is the most col can punish me.
  (col knows my strategy)

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$\max_p \min_q (v_R(p,q))$. i choose the p with col's least punishing move

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· so... uh... how do i calculate this?

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for every possible value of p, what is the most punishing move?

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$\vec{p} = (p, 1-p)$

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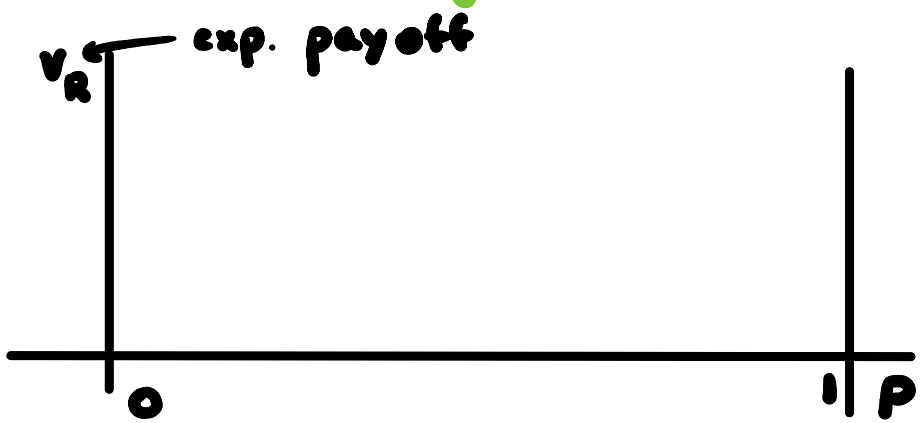

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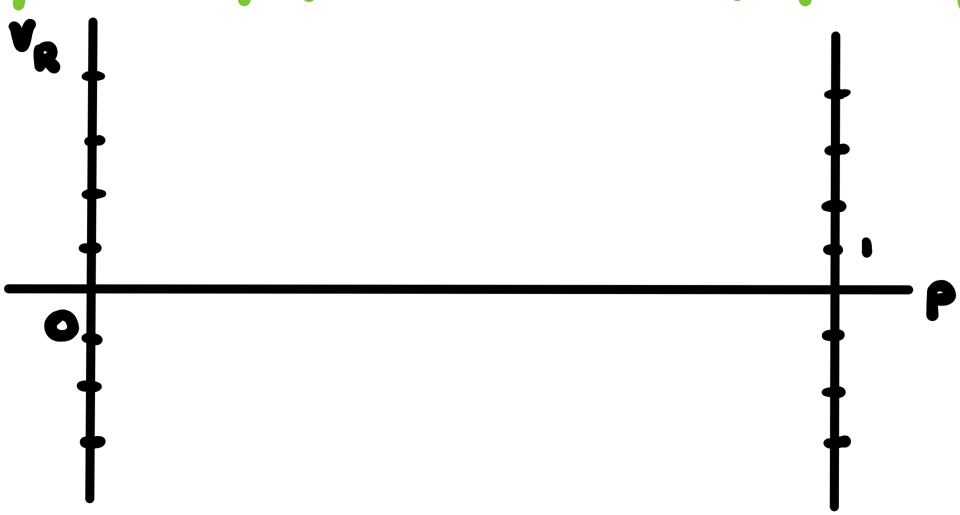
for every possible value of $\vec{p}$, what are the expected payoffs for every pure $\vec{q}$?



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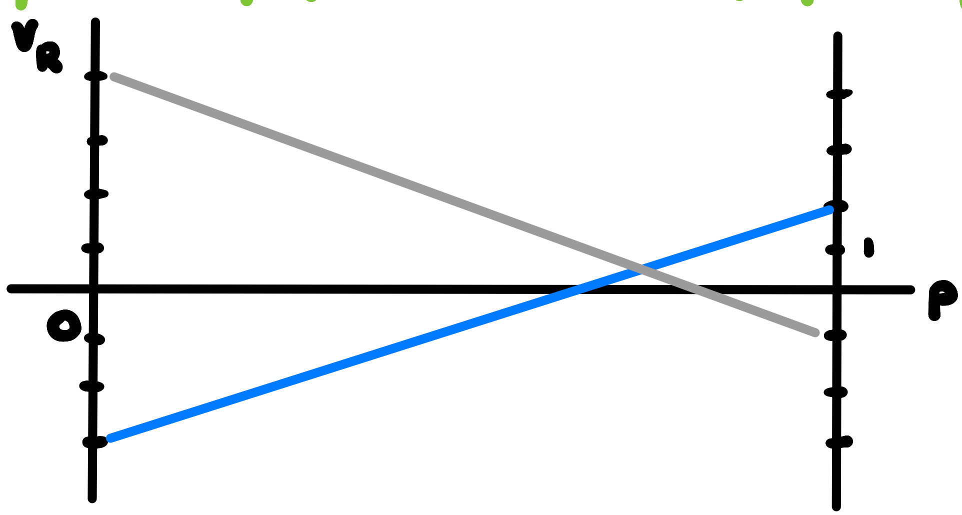

A: $5p - 3$

B: $4 - 5p$

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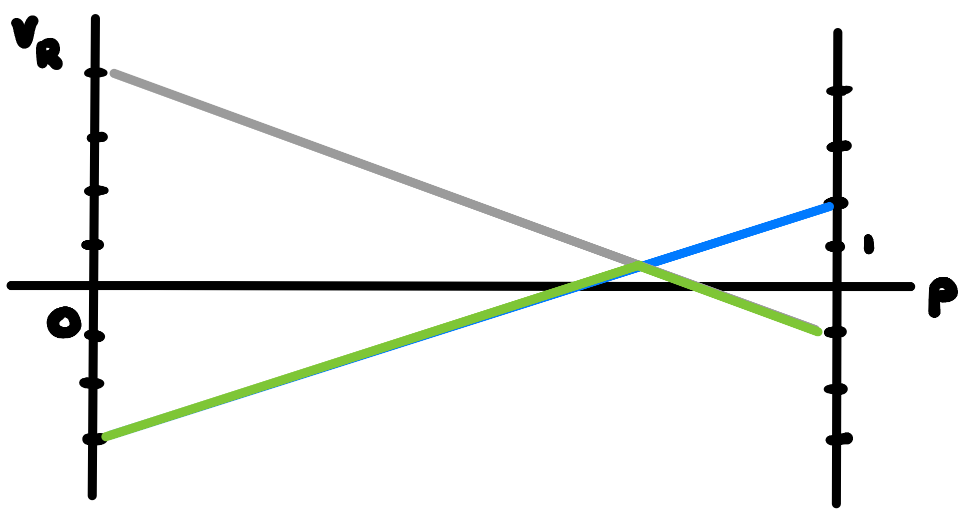

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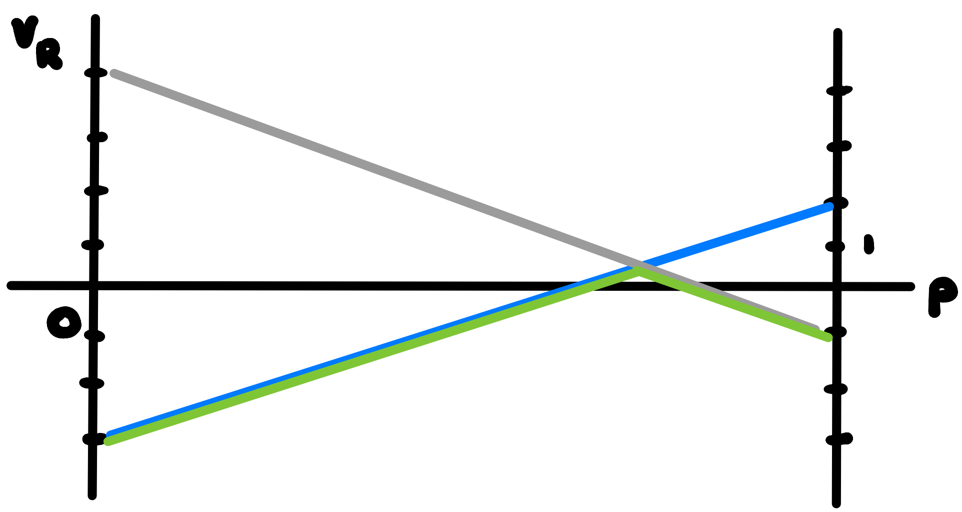

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$\min_q (V_R(p,q))$

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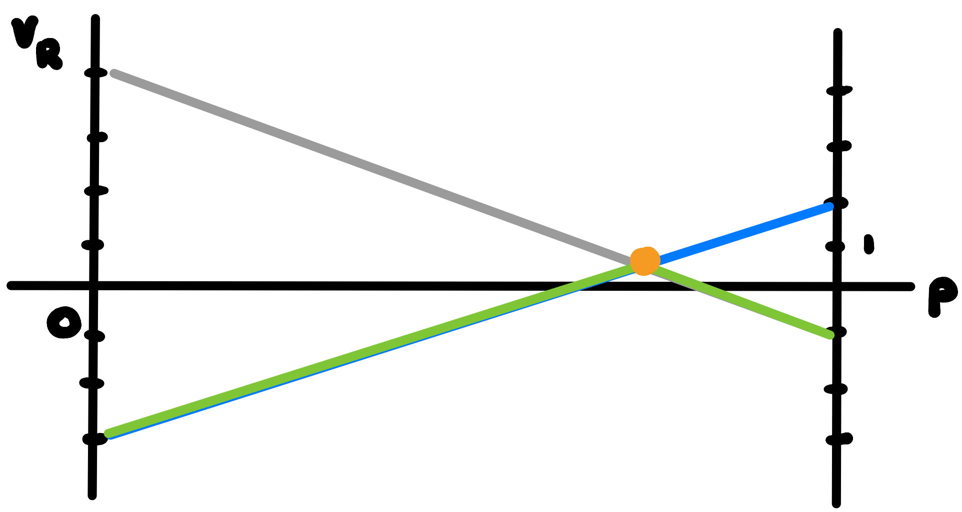

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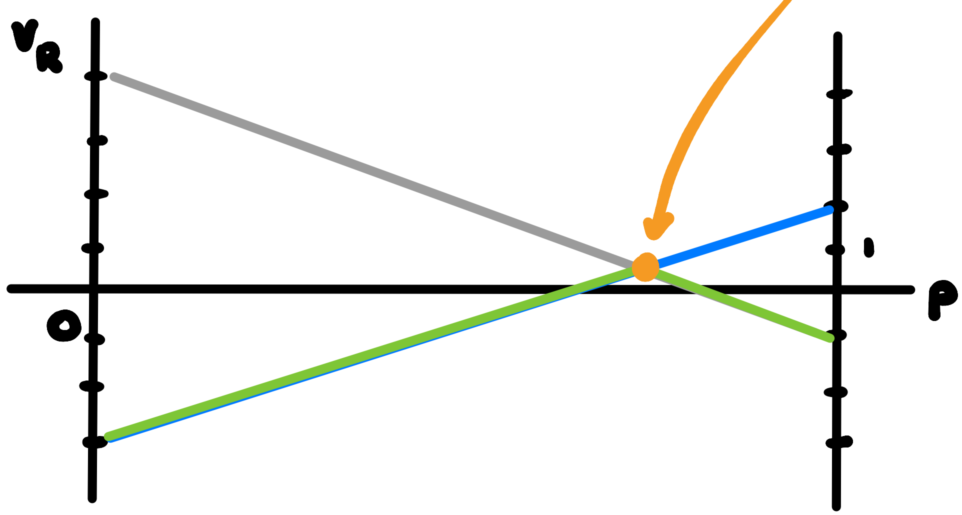

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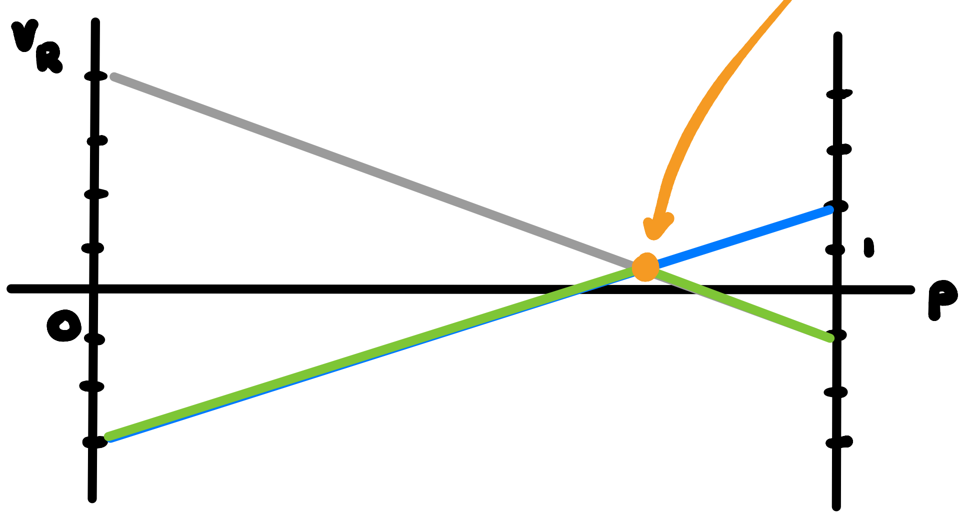

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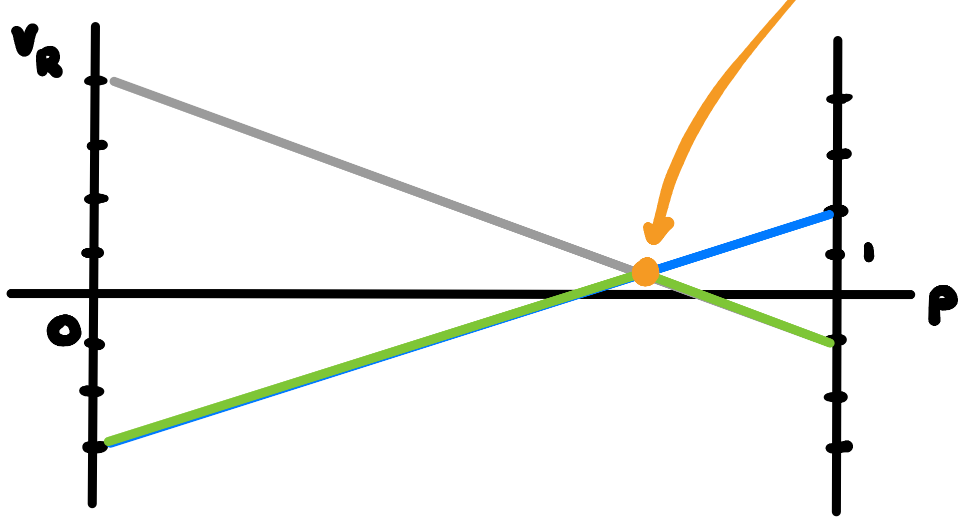

VALUE

$p = \left(\frac{7}{10}, \frac{1}{2}\right)$

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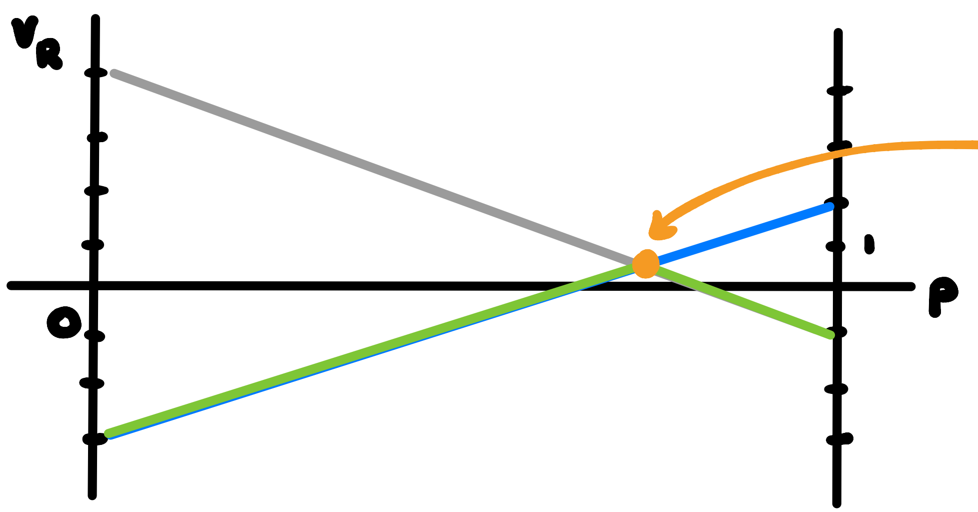

$\left(\frac{7}{10}, \frac{1}{2}\right)$

$p \quad v$

$\Rightarrow \vec{p} = \left(7/10,\ 3/10\right)$

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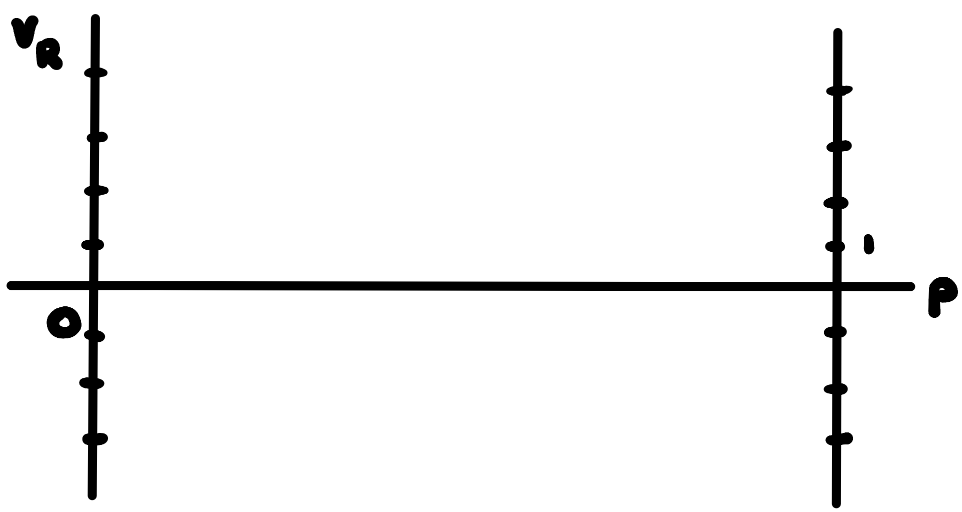

$\left(\frac{7}{10}, \frac{1}{2}\right)$

p v

$\Rightarrow \vec{q}$?

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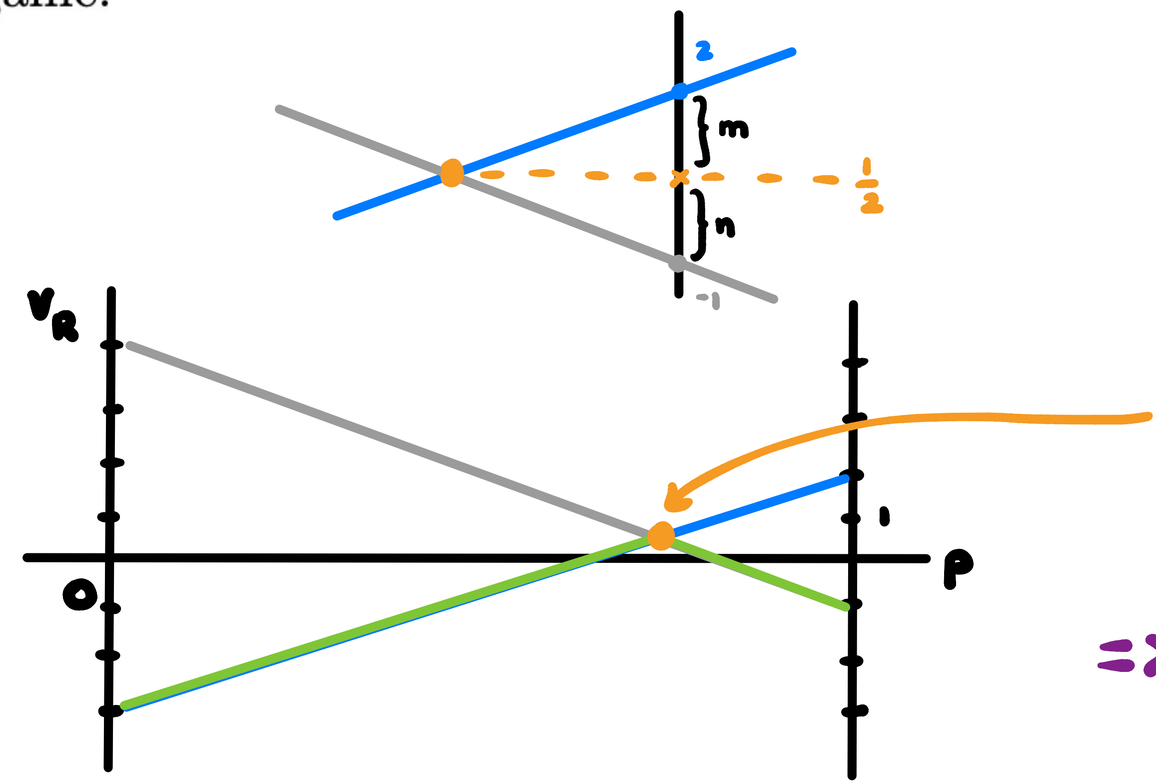


$\left(\frac{7}{10}, \frac{1}{2}\right)$

p v

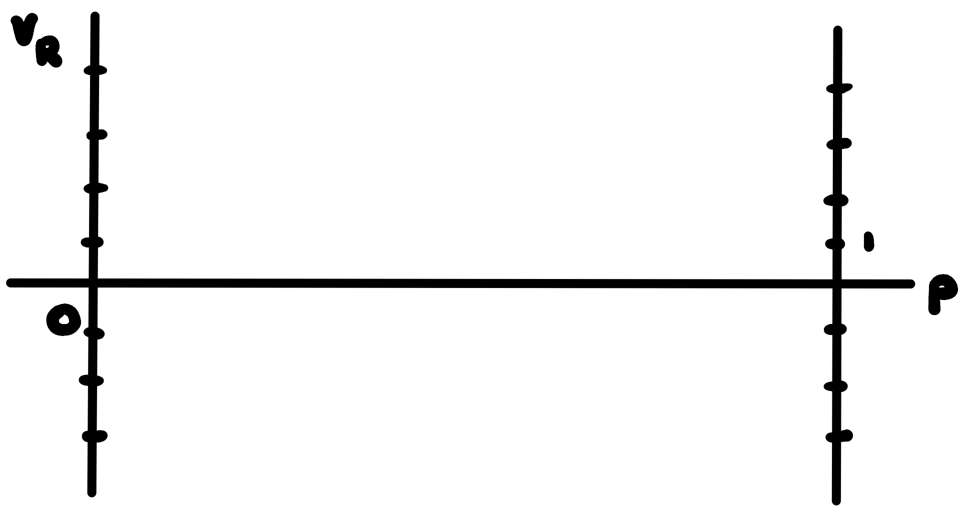
$$\Rightarrow \vec{q} = \left(\frac{n}{m+n}, \frac{m}{n+m}\right)$$

1 ZERO-SUM GAMES

(b) Now consider the game with payoffs:

		column player	
		A	**B**
row	**1**	$(-\frac{1}{2}, \frac{1}{2})$	$(-1, 1)$
player	**2**	$(1, -1)$	$(\frac{2}{3}, -\frac{2}{3})$

- now, it's your turn....



I ZERO-SUM GAMES

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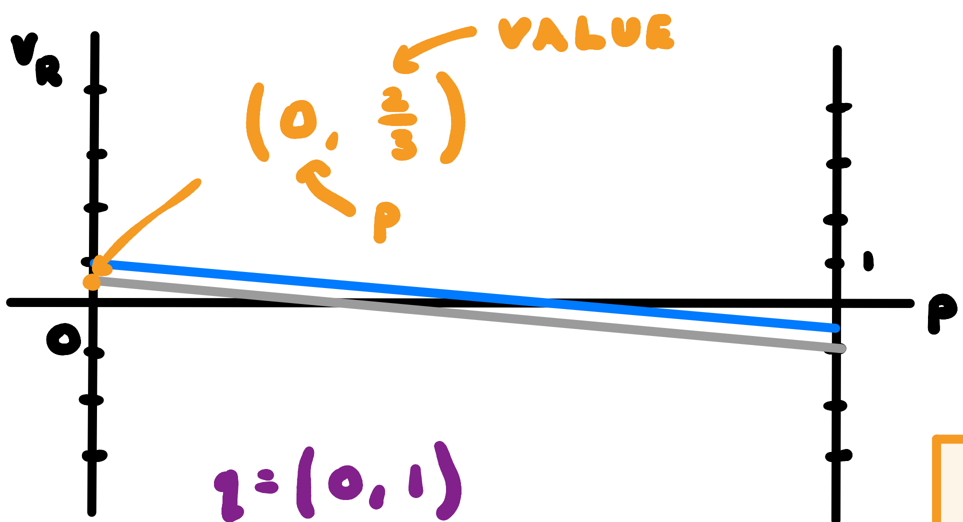

A: $1 - \frac{3}{2}p$

B: $\frac{2}{3} - \frac{5}{3}p$

$\boxed{\text{DOMINANT STRATEGY}}$

3 VERTEX-COVER

3. **(Duality in graph problems)** Recall that a *vertex cover* of a graph $G = (V, E)$ is a subset of the vertices such that every edge in E is adjacent to at least one of the vertices in the subset. The minimum vertex cover is a vertex cover with the fewest possible vertices.

* this problem is NP-Complete.
(a proof of this is in 15-251)

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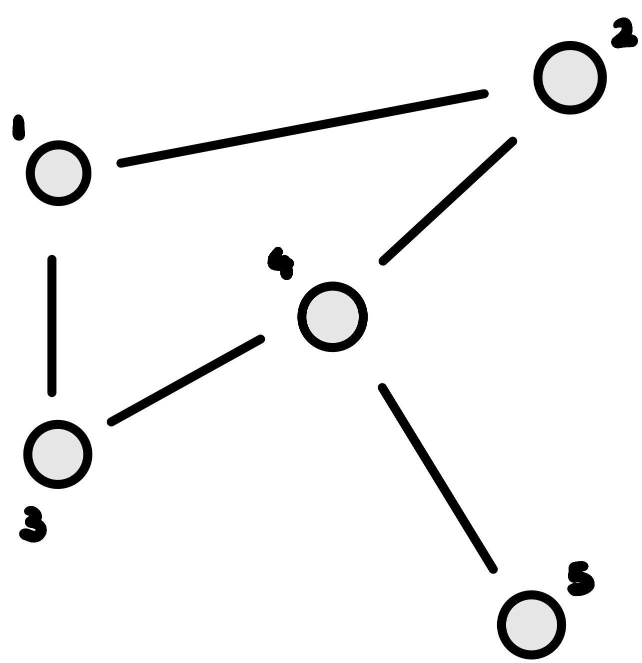

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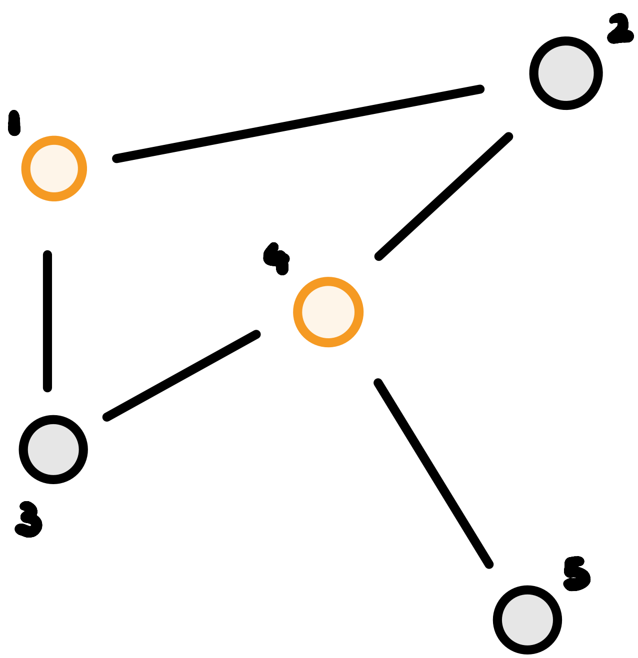

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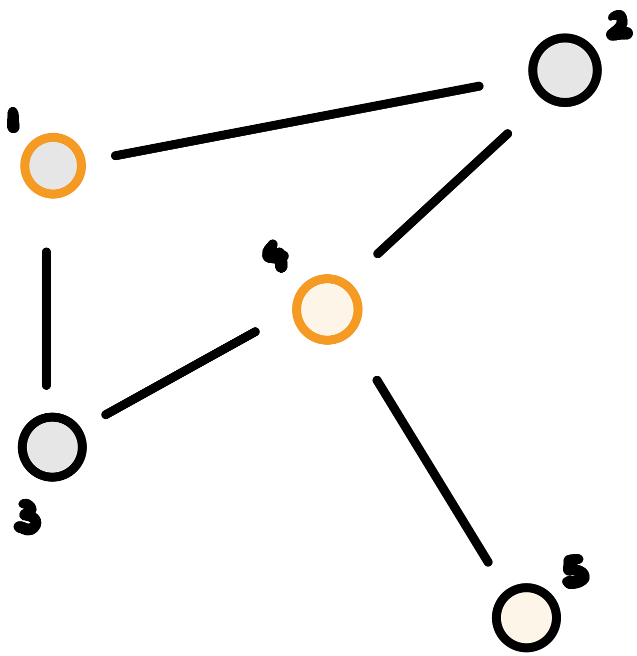

- **variables:**
  (what decisions do we need to make)

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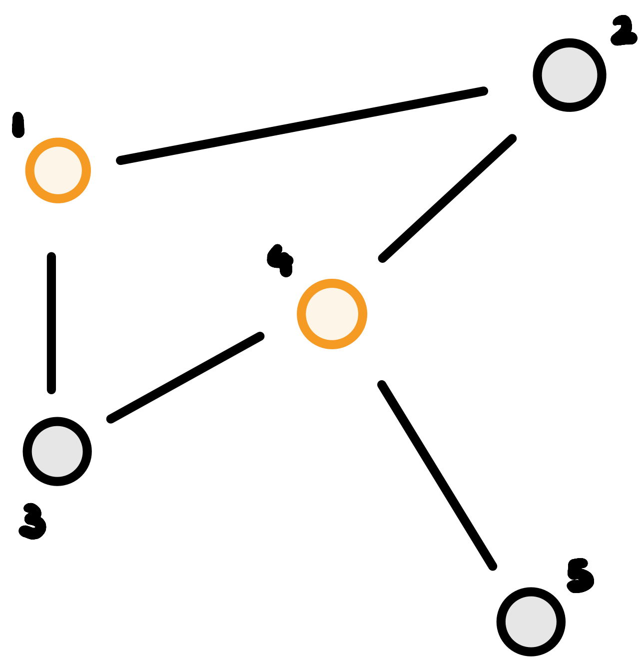

$w_i = 1$ if in cover

$w_i = 0$ if not in cover

- variables: w_1, w_2, w_3, w_4, w_5

(what decisions do we need to make)

- max/min: ______________

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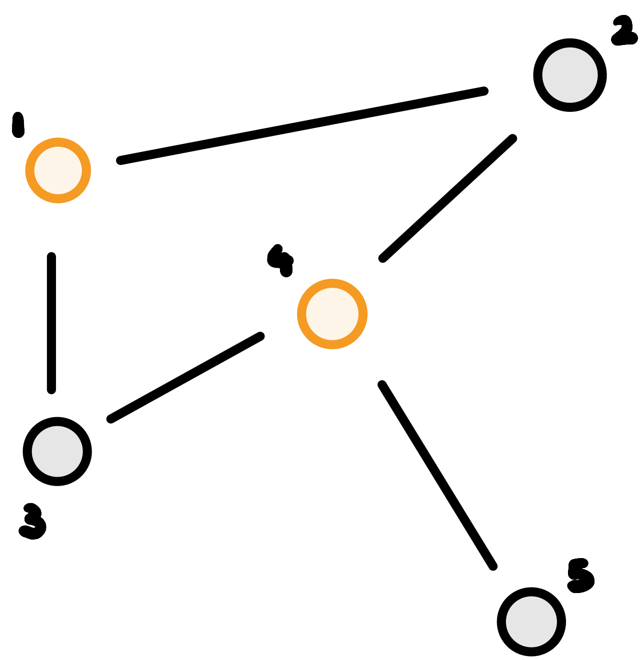

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- variables: w_1, w_2, w_3, w_4, w_5
  (what decisions do we need to make)
- max/min: $\sum w_i$ ← $i \in [1, 5]$
- subject to:
  (requirement of a vertex cover)

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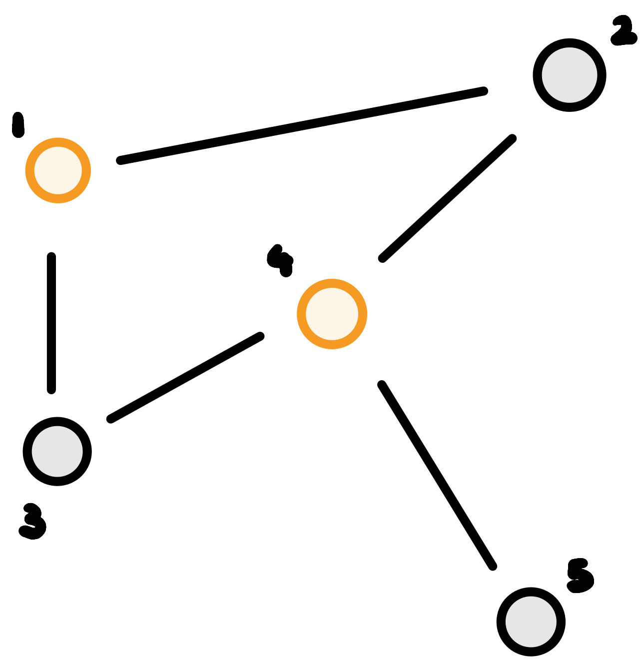

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- variables: w_1, w_2, w_3, w_4, w_5
  (what decisions do we need to make)

- max / min : $\sum w_i \leftarrow i \in [1, 5]$

- subject to:
  (requirement of a vertex cover)

$* w_i \geq 0$

$w_1 + w_2 \geq 1$ ← one requirement for each edge
$w_1 + w_3 \geq 1$
$w_2 + w_4 \geq 1$
$w_3 + w_4 \geq 1$
$w_4 + w_5 \geq 1$

we solved an NP-complete problem! SIM!!!

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who said an LP alg must adhere to this?

$w_i = 1$ if in cover
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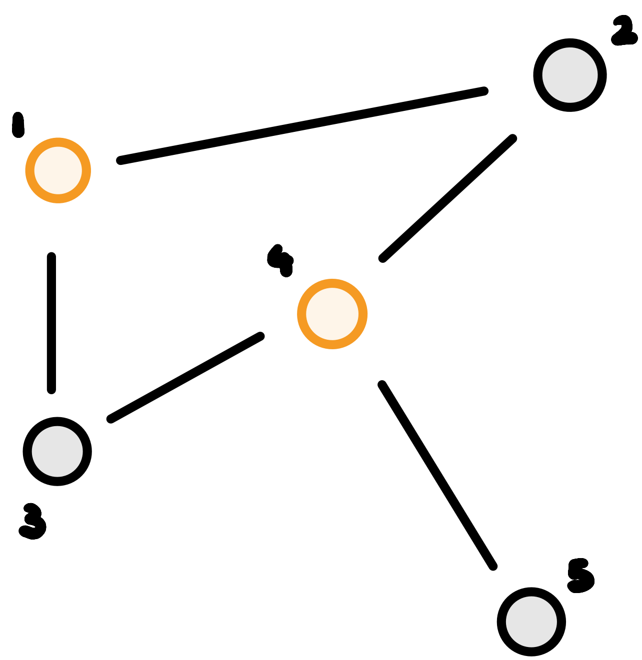

- variables: w_1, w_2, w_3, w_4, w_5
  (what decisions do we need to make)
- max/min: $\sum w_i$ ← $i \in [1, 5]$
- subject to:
  (requirement of a vertex cover)

$w_1 + w_2 \geq 1$ ← one requirement for each edge
$w_1 + w_3 \geq 1$
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we ~~solved~~ an NP-complete problem! SIM!!!

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(b) Write down the dual of this LP. What does it mean?

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minimize $w_1 + w_2 + w_3 + w_4 + w_5$

s.t. $w_1 + w_2 \geq 1$

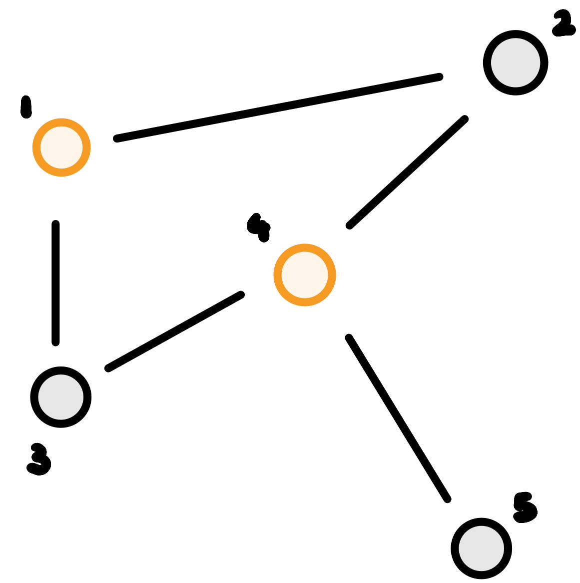
$w_1 + w_3 \geq 1$

$w_2 + w_4 \geq 1$

$w_3 + w_4 \geq 1$

$w_4 + w_5 \geq 1$

$w_i \geq 0 \ (\forall i \in [5])$

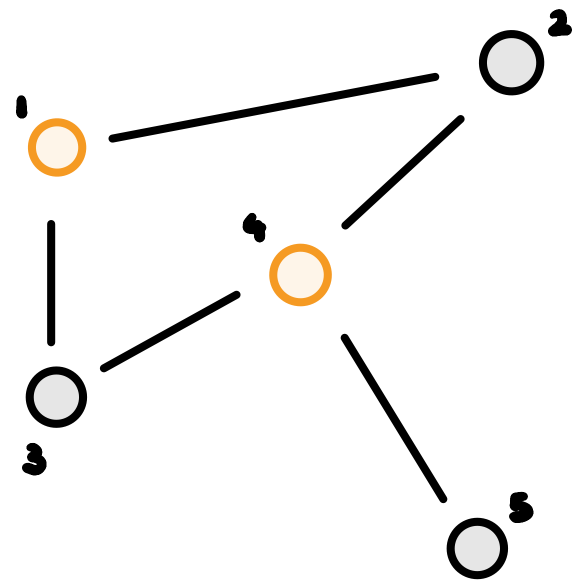


3 VERTEX-COVER

(b) Write down the dual of this LP. What does it mean?

$$\text{minimize } w_1 + w_2 + w_3 + w_4 + w_5$$

$$\begin{aligned} \text{s.t. } & w_1 + w_2 \geq 1 \\ & w_1 + w_3 \geq 1 \\ & w_2 + w_4 \geq 1 \\ & w_3 + w_4 \geq 1 \\ & w_4 + w_5 \geq 1 \\ & w_i \geq 0 \ (\forall i \in [5]) \end{aligned}$$



WRITE INTO MATRIX FORM
(w/ a small example)

3 VERTEX-COVER

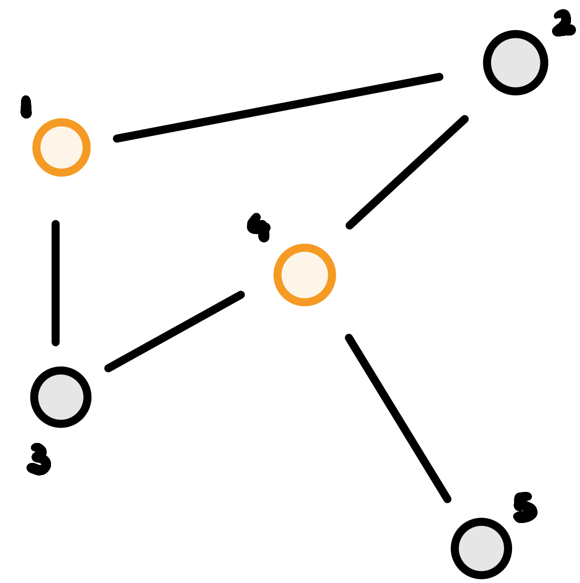
(b) Write down the dual of this LP. What does it mean?

$$\text{minimize } \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{bmatrix}$$

s.t.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{bmatrix} \geq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

* $w_i \geq 0$



WRITE INTO MATRIX FORM
(w/ a small example)

* write out:
  - rows correspond to
  - cols correspond to

3 VERTEX-COVER

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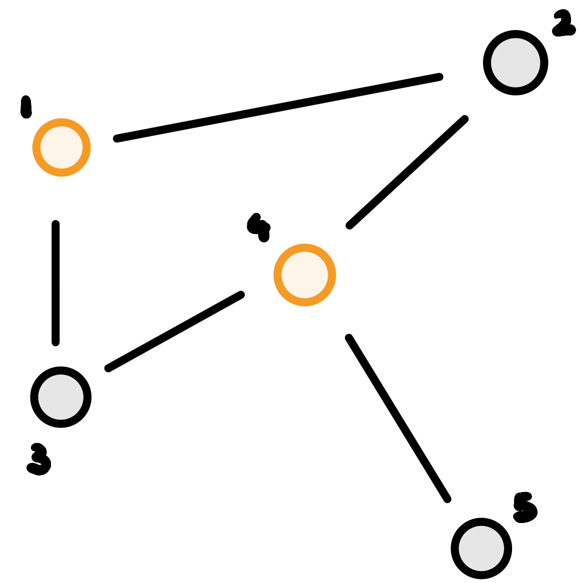
s.t.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{bmatrix} \geq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$



WRITE INTO MATRIX FORM
(w/ a small example)

CONVERT VIA FORMULA

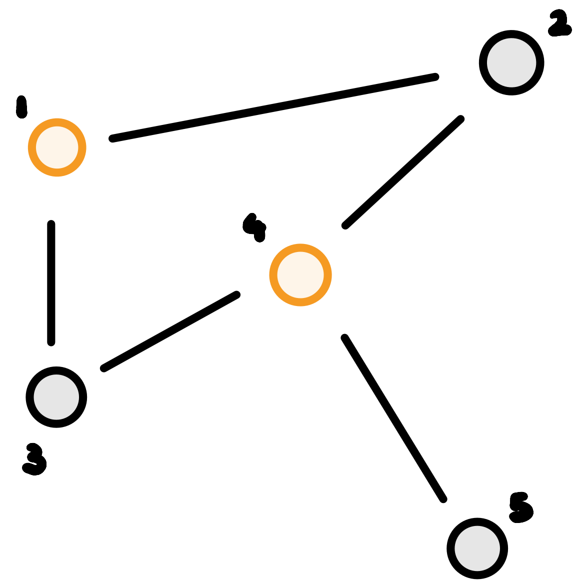
3 VERTEX-COVER

(b) Write down the dual of this LP. What does it mean?

$$\text{maximize } \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

s.t.

$$\begin{bmatrix} y_1 & y_2 & y_3 & y_4 & y_5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$



WRITE INTO MATRIX FORM
(w/ a small example)

CONVERT VIA FORMULA

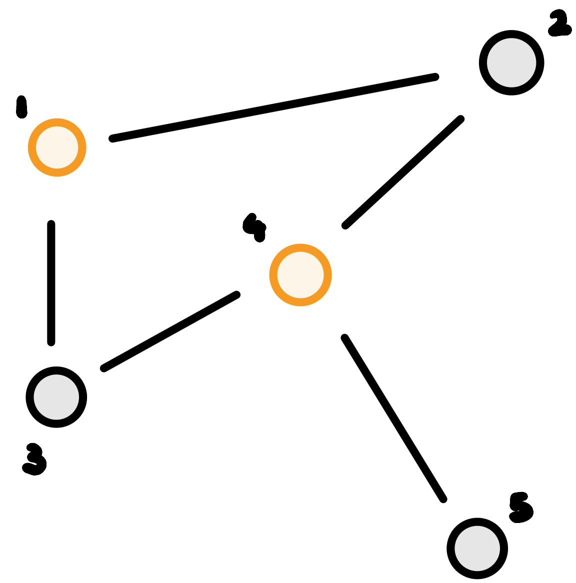
3 VERTEX-COVER

(b) Write down the dual of this LP. What does it mean?

$$\text{maximize} \begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\\ y_4\\ y_5\end{bmatrix}$$

s.t.

$$\underset{\text{EDGES}}{\begin{bmatrix}y_1 & y_2 & y_3 & y_4 & y_5\end{bmatrix}} \overset{\text{VERTICES}}{\begin{bmatrix}1 & 1 & 0 & 0 & 0\\ 1 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 1 & 0\\ 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1 & 1\end{bmatrix}}\ \text{EDGES} \le \begin{bmatrix}1\\ 1\\ 1\\ 1\\ 1\end{bmatrix}$$



WRITE INTO MATRIX FORM
(w/ a small example)

CONVERT VIA FORMULA

$y_1 + y_2 \le 1$

(edge 1) + (edge 2) $\le$ 1

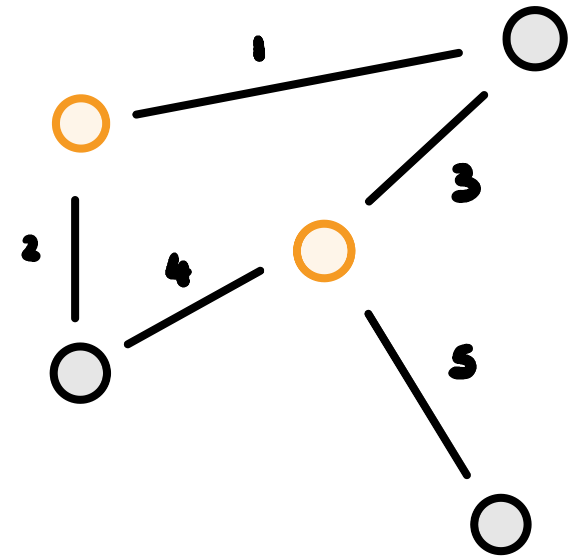
* you can only select at most one edge

3 VERTEX-COVER

(b) Write down the dual of this LP. What does it mean?

$$\text{maximize} \quad y_1 + y_2 + y_3 + y_4 + y_5$$

$$\begin{aligned}
\text{s.t.} \quad & y_1 + y_2 \le 1 \\
& y_1 + y_3 \le 1 \\
& y_2 + y_4 \le 1 \\
& y_3 + y_4 + y_5 \le 1 \\
& y_5 \le 1 \\
& y_i \ge 0 \ (\forall i \in [5])
\end{aligned}$$



3 VERTEX-COVER

(c) Based on your answer to Part (a), which of the following are true, and why?

- ☐ For any graph G, size of minimum vertex cover $=$ size of maximum matching
- ☐ For any graph G, size of minimum vertex cover $\leq$ size of maximum matching
- ☐ For any graph G, size of minimum vertex cover $\geq$ size of maximum matching