The Algorithmic Magic of Polynomials

David Woodruff

Polynomials

- Polynomial: $p(x) = c_d x^d + c_{d-1} x^{d-1} + \cdots + c_1 x + c_0$
- $(c_d, c_{d-1}, \ldots, c_0)$ completely describes p
- Addition: $(x^2 + 2x - 1) + (3x^3 + 7x) = 3x^3 + x^2 + 9x - 1$
- Multiplication:
  $$(x^2 + 2x - 1) \cdot (3x^3 + 7x) = 3x^5 + 4x^3 + 6x^4 + 14x^2 - 7x$$
- Evaluation: p(5) = $c_d 5^d + c_{d-1} 5^{d-1} + \cdots + c_1 5 + c_0$

Evaluating a Polynomial Quickly

- Polynomial: $p(x) = c_d x^d + c_{d-1} x^{d-1} + \cdots + c_1 x + c_0$

- Evaluate at a point b in time O(d) using Horner's Rule:

- Compute: c_d

  $c_{d-1} + c_d \cdot b$

  $c_{d-2} + c_{d-1} \cdot b + c_d \cdot b^2$

  …

- Each step has O(1) operations – multiply by and add coefficient

Polynomial Degree

- Polynomial: $p(x) = c_d x^d + c_{d-1} x^{d-1} + \cdots + c_1 x + c_0$
- If $c_d \neq 0$, the degree is d
- If A(x) has degree d and B(x) has degree d, then A(x) + B(x) has degree at most d

Why is the degree at most d?

Roots of Polynomials

- A root of a polynomial is a number r for which A(r) = 0
- Fundamental theorem of algebra: a non-zero degree-d polynomial has at most d roots
  - Implies any distinct degree d polynomials A(x) and B(x) can evaluate to the same value on at most d different values x. Why?
  - A(x) – B(x) has degree at most d, so can have at most d roots
  - A degree d polynomial is determined by its evaluations on d+1 distinct points $x_0, \ldots, x_d$
- Given $(x_0, y_0), \ldots, (x_d, y_d)$ for distinct $x_0, \ldots, x_d$, is there a polynomial p of degree at most d with $p(x_i) = y_i$ for each i?

Unique Reconstruction Theorem

- Given $(x_0, y_0), \ldots, (x_d, y_d)$ for distinct $x_0, \ldots, x_d$, there exists a polynomial of degree at most d for which $p(x_i) = y_i$ for each i
- Define $R_i(x) = \prod_{j \neq i}(x - x_j) \,/\, \prod_{j \neq i}(x_i - x_j)$, which has degree d
- $R_i(x_j) = 0$ for $j \neq i$
- $R_i(x_i) = 1$
- $p(x) = \sum_{i=0,\ldots,d} y_i \cdot R_i(x)$

Example of Polynomial Reconstruction

- Given pairs (5,1), (6,2), and (7,9), we would like to find a degree-2 polynomial that passes through these points

- $R_0(x) = \frac{(x-6)(x-7)}{(5-6)(5-7)} = \frac{1}{2}(x-6)(x-7)$

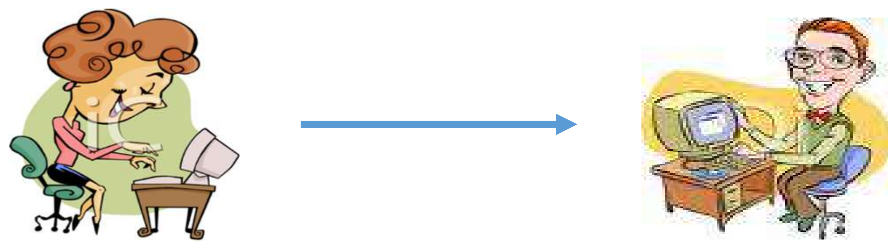
- $R_1(x) = \frac{(x-5)(x-7)}{(6-5)(6-7)} = -(x-5)(x-7)$

- $R_2(x) = \frac{(x-5)(x-6)}{(7-5)(7-6)} = \frac{1}{2}(x-5)(x-6)$

- $p(x) = 1 \cdot R_0(x) + 2 \cdot R_1(x) + 9 \cdot R_2(x) = 3x^2 - 32x + 86$

Polynomials For Error Correcting Codes

A Deletion Channel



5, 19, 2, 3, 2 *, 19, *, *, 2

- Alice has d+1 numbers and wants to send them to Bob
- Up to k of the numbers might be replaced with a *
- *How can Bob learn Alice's numbers?*

A Deletion Channel

- Alice could repeat each number k+1 times

- If k = 3, she sends:

  5, 5, 5, 5, 19, 19, 19, 19, 2, 2, 2, 2, 3, 3, 3, 3, 2, 2, 2, 2

- This is (d+1)(k+1) words of communication

- *Can we get d+k+1 communication?*

A Deletion Channel

- Suppose Alice has $c_d, c_{d-1}, c_{d-2,\ldots,}\ c_0$
- She interprets these as the coefficients of a polynomial P(x):

$$P(x) = \sum_{i=0,\ldots,d} c_i x^i$$

- Alice sends P(0), P(1), P(2), …, P(d+k)
- Bob gets at least d+1 of these numbers. By the unique reconstruction theorem, he recovers P(x), and hence $c_d, c_{d-1}, c_{d-2,\ldots,}\ c_0$

General Error Correction

- Now the adversary can replace up to k numbers with other numbers
- If Alice wants to send Bob a single number x, how many times does she need to copy it?
  - 2k+1, to ensure the majority symbol is correct
- Now Alice has $c_d, c_{d-1}, c_{d-2,\ldots,}\ c_0$, which she writes as a polynomial $P(x) = \sum_{i=0,\ldots,d} c_i x^i$
- Suppose Alice sends P(0), P(1), …, P(r). How large does r need to be?
  - d+2k+1 points is enough, so r = d+2k
  - If it weren't, there'd be another degree at most d polynomial Q agreeing on d+k+1 of these evaluations, so P and Q would agree on at least d+1 points. A contradiction

Algorithm for General Error Correction

- But how to find P(x) given k corruptions to P(0), P(1), …, P(d+2k)?
- Suppose Bob receives $r_0, r_1, \ldots, r_{d+2k}$
- Z = {i such that $r_i \neq P(i)$} , and so $|Z| \leq k$
- $E(x) = \prod_{i \in Z}(x - i)$
- $P(x) \cdot E(x) = r_x \cdot E(x)$ for all x = 0, 1, 2, …, d+2k

Berlekamp-Welch Algorithm

- $P(x) \cdot E(x) = r_x \cdot E(x)$ for all x = 0, 1, 2, …, d+2k (*)
- $E(x) = x^k + e_{k-1}x^{k-1} + e_{k-2}x^{k-2} + \cdots + e_0$ if degree(E(x)) = k
- $P(x) \cdot E(x) = f_{d+k}x^{d+k} + f_{d+k-1}\, x^{d+k-1} + \cdots + f_0$
- Plugging each x = 0, 1, 2, …, d+2k into (*), we get a linear equation relating $f_{d+k,}\, f_{d+k-1}, \ldots, f_0, e_{k-1,} e_{k-2}, \ldots, e_0$
- d+2k+1 unknowns and d+2k+1 equations
- Equations are linearly independent, so get $(P(x) \cdot E(x))$ and E(x), output $\frac{(P(x) \cdot E(x))}{E(x)}$

Polynomials for Finding Maximum Matchings

Multivariate Polynomials

- $p(x_1, x_2, x_3, x_4) = x_1 x_2^2 x_4 + x_3 x_4^2 + x_1 x_2^2 x_3^2 x_4$
- Degree of monomial $x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4}$ is $i_1 + i_2 + i_3 + i_4$
- Degree of p is the maximum degree of any of its monomials

Schwartz-Zippel Lemma for Multivariate Polynomials

- [Schwartz-Zippel] Let P($X_1, \ldots, X_m$) be a non-zero, m-variable, degree at most d polynomial, and let S be a subset from the field F. If each X_i is chosen independently in S,

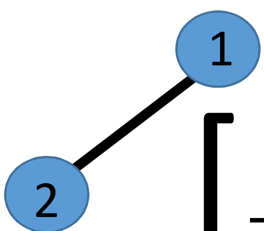
$$\Pr[P(X_1, \ldots, X_m) = 0] \leq \frac{d}{|S|}$$

- Sanity check: if m = 1, a non-zero degree-d polynomial has at most d roots
- If |F| > 3d, how can we tell if P is the all zeros polynomial w.pr. 2/3?
- Choose $X_1, \ldots, X_m$ independently from F, and evaluate $P(X_1, \ldots, X_m)$

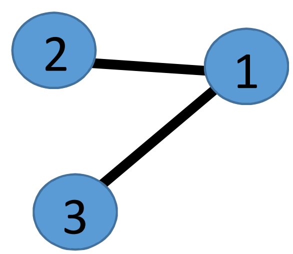
Tutte Matrix

- If G is a graph on vertices $v_1, \ldots, v_n$, the Tutte matrix is a |V| x |V| matrix M(G) with

$$M(G)_{i,j} = \begin{cases} x_{i,j} & if \; \{v_i, v_j\} \in E \; and \; i < j \\ -x_{j,i} & if \; \{v_i, v_j\} \in E \; and \; i > j \\ 0 & if \; (v_i, v_j) \notin E \end{cases}$$



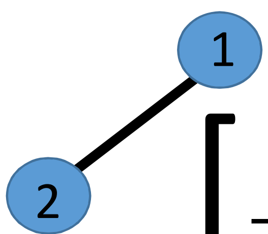
$$\begin{bmatrix} 0 & x_{12} \\ -x_{12} & 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 & x_{12} & x_{13} \\ -x_{12} & 0 & 0 \\ -x_{13} & 0 & 0 \end{bmatrix}$$

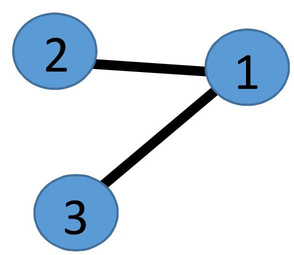
Tutte Determinant Theorem

- [Tutte] A graph has a perfect matching if and only if the determinant of M(G) is not the zero polynomial (a matching is perfect if all nodes are matched)



$$\begin{bmatrix} 0 & x_{12} \\ -x_{12} & 0 \end{bmatrix}$$

$$\det\big(M(G)\big) = x_{12}^2$$



$$\begin{bmatrix} 0 & x_{12} & x_{13} \\ -x_{12} & 0 & 0 \\ -x_{13} & 0 & 0 \end{bmatrix}$$

$$\det\big(M(G)\big) = 0$$

- det(M(G)) is a polynomial of degree at most n, and could have n! terms
- *How can we determine if G has a perfect matching with probability at least 2/3?*
- Choose a field F with |F| > 3n, randomly fill in the $x_{i,j}$ values, and compute determinant!

Finding a Perfect Matching

- We can quickly determine if G has a perfect matching
- Can reduce the error probability to $1/n^3$, say, by choosing $|F| = n^4$
- But how to output the edges in the perfect matching?
- For each edge e,
  - Remove e and see if there is still a perfect matching
  - If there is no perfect matching, put e back in G, otherwise discard e
- At the end, will be left with exactly n/2 edges in a perfect matching

Finding a Maximum Matching

- Can we find a maximum matching if we can find a perfect matching?
- Given a graph G, connect n-2k new nodes to every node in G
- If G has a matching of size at least k, then this new graph has a perfect matching
- If the maximum matching size of G is less than k, then this new graph does not have a perfect matching
- Binary search on k