Game Theory and Lower Bounds for Randomized Algorithms

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Outline

- 2-player zero-sum games and minimax optimal strategies

- Connection to randomized algorithms

- General sum games, Nash equilibria

Game Theory

- How people make decisions in social and economic interactions
  - Applications to computer science
- Users interacting with each other in large systems
  - Routing in large networks
  - Auctions on Ebay

Definitions

- A game has
  - Participants, called players
  - Each player has a set of choices, called actions
  - Combined actions of players leads to payoffs for each player

Shooter-Goalie Game

- 2 players: shooter and goalie
- Shooter has 2 actions: shoot to her left or shoot to her right
- Goalie has two actions: dive to shooter's left or to shooter's right
    - left and right are defined with respect to shooter's actions
- Set of actions for both Shooter and Goalie is {L, R}
- If shooter and goalie each choose L, or each choose R, then goalie makes a save
- If shooter and goalie choose different actions, then the shooter makes a goal

Payoff Matrix

- If goalie makes a save, goalie has payoff +1, shooter has payoff -1
- If shooter makes a goal, goalie has payoff -1, shooter has payoff +1

payoff matrix M	goalie L	goalie R
shooter L	$(-1, 1)$	$(1, -1)$
R	$(1, -1)$	$(-1, 1)$

- Payoff is (r,c), where r is payoff to row player, and c is payoff to the column player
- For each entry (r,c), r+c = 0. This is called a zero-sum game
- Zero-sum game does not imply “fairness”. If all entries are (1,-1) it is still zero-sum

An Aside

- Row-payoff matrix R consists of the payoffs to the row player
- C is the column-payoff matrix
- $\mathrm{M}_{\mathrm{i,j}} = (\mathrm{R}_{\mathrm{i,j}}, \mathrm{C}_{\mathrm{i,j}})$ for all i and j

payoff matrix M	goalie	
	L	R
shooter L	$(-1, 1)$	$(1, -1)$
R	$(1, -1)$	$(-1, 1)$

Row payoff matrix	goalie	
	L	R
shooter L	-1	1
R	1	-1

- R + C = 0 for zero-sum games

Pure and Mixed Strategies

- How should the players play?
- Pure strategy:
  - Row player chooses a deterministic action I
  - Column player chooses a deterministic action J
  - Payoff is $R_{I,J}$ for row player, and $C_{I,J}$ for column player
- Pure strategies are deterministic, what about randomized strategies?
  - Players have a distribution over their actions
  - Row player decides on a $p_i \in [0,1]$ for each row, with $\sum_{\text{actions } i} p_i = 1$
  - Column player decides on a $q_j \in [0,1]$ for each column, with $\sum_{\text{actions } j} q_j = 1$
- Distributions p and q are mixed strategies

How to define payoff for mixed strategies?

Expected Payoff

- Assume players have independent randomness
- $V_R(p,q) = \sum_{i,j} \Pr[\text{row player plays i, column player plays j}] \cdot R_{i,j} = \sum_{i,j} p_i q_j R_{i,j}$
- $V_C(p,q) = \sum_{i,j} \Pr[\text{row player plays i, column player plays j}] \cdot C_{i,j} = \sum_{i,j} p_i q_j C_{i,j}$
- What is $V_R(p,q) + V_C(p,q)$?
  - 0, since zero-sum game

payoff matrix M		goalie L	goalie R
shooter	L	$(-1,1)$	$(1,-1)$
	R	$(1,-1)$	$(-1,1)$

If p = (.5, .5) and q = (.5, .5) what is V_R?

$$V_R = .25 \cdot (-1) + .25 \cdot 1 + \ .25 \cdot 1 + \ .25 \cdot (-1)$$

If p = (.75, .25) and q = (.6, .4) what is V_R?

$$V_R = -0.1$$

Minimax Optimal Strategies

- Row player wants a distribution p^* maximizing her expected payoff over all strategies q of her opponent
- p^* achieves lower bound lb = $\max\limits_{\mathrm{p}} \min\limits_{\mathrm{q}} \mathrm{V_R}(\mathrm{p}, \mathrm{q})$

$$\mathrm{lb} := \overbrace{\max_{\mathbf{p}} \underbrace{\min_{\mathbf{q}} V_R(\mathbf{p}, \mathbf{q})}_{\text{payoff when opponent plays optimal strategy against our choice } \mathbf{p}}}^{\text{mixed strategy that maximizes the minimum expected payoff}}$$

- The row player can guarantee this expected payoff no matter what the column player does. lb is a lower bound on the row-player’s payoff

Minimax Optimal Strategies

- Column player wants distribution q^* maximizing his expected payoff over all strategies p of his opponent
  - q^* achieving $\max_q \min_p V_C(p, q)$

- Claim: $\max_q \min_p V_C(p, q) = -\min_q \max_p V_R(p, q)$
- Proof: $\max_q \min_p V_C(p, q) = \max_q \min_p -V_R(p, q)$

$$= \max_q(-\max_p V_R(p, q))$$

$$= -\min_q \max_p V_R(p, q)$$

Payoff to row player if column player plays q^* is ub = $\min_q \max_p V_R(p, q)$

Column player can guarantee the row player does not achieve a larger expected payoff, so this is an upper bound ub on row player's expected payoff

Lower and Upper Bounds

- Row player guarantees she has expected payoff at least

$$\text{lb} = \max_{p} \min_{q} V_R(p, q)$$

- Column player guarantees row player has expected payoff at most

$$\text{ub} = \min_{q} \max_{p} V_R(p, q)$$

lb ≤ ub, but how close is lb to ub?

A Pure Strategy Observation

- Suppose we want to find row player's optimal strategy p^*

- Claim: can assume column player plays a pure strategy. Why?
  - For any strategy p of the row player, $V_R(p,q) = \sum_{i,j} p_i q_j R_{i,j} = \sum_j q_j \cdot (\sum_i p_i R_{i,j})$
  - Column player can choose q to be the j for which $\sum_i p_i R_{i,j}$ is minimal

- lb = $\max_p \min_q V_R(p,q) = \max_p \min_j \sum_i p_i R_{i,j}$

- ub = $\min_q \max_p V_R(p,q) = \min_q \max_i \sum_j q_j R_{i,j}$

Shooter-Goalie Example

payoff matrix M	goalie L	goalie R
shooter L	$(-1,1)$	$(1,-1)$
R	$(1,-1)$	$(-1,1)$

Claim: minimax-optimal strategy for both players is (.5, .5)

Proof: For the shooter (row-player), let $\mathbf{p} = (p_1, p_2)$ be the minimax optimal strategy

$p_1 \geq 0, p_2 \geq 0$, and $p_1 + p_2 = 1$. Write **p** = (p, 1-p) with p in [0,1]

Suppose goalie (column-player) plays L

Shooter's payoff is $p \cdot (-1) + (1-p) \cdot (1) = 1 - 2p$

Suppose goalie plays R

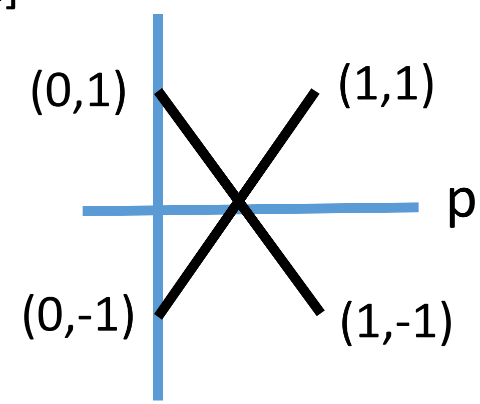
Shooter's payoff is $p \cdot (1) + (1-p) \cdot (-1) = 2p - 1$

Choose p ∈ [0,1] to maximize lb = $\max_p \min(1-2p, 2p-1)$

p = ½ realizes this, and lb = 0

Similarly show optimal strategy **q**$= (q_1, q_2)$ of goalie is (1/2,1/2) and ub = 0

ub = lb = 0, which is the *value of the game*



Asymmetric Shooter-Goalie

	L	R
shooter L	$(-\frac{1}{2}, \frac{1}{2})$	$(1, -1)$
R	$(1, -1)$	$(-1, 1)$

Goalie is now weaker on the left

Let **p** = $(\mathrm{p}_1, \mathrm{p}_2)$ be the minimax optimal shooter (row-player) strategy

Suppose goalie (column player) plays L

Shooter's payoff is $\mathrm{p} \cdot \left(-\frac{1}{2}\right) + (1-\mathrm{p}) \cdot (1) = 1 - (\frac{3}{2})\mathrm{p}$

Suppose goalie plays R

Shooter's payoff is $\mathrm{p} \cdot (1) + (1-\mathrm{p}) \cdot (-1) = 2\mathrm{p} - 1$

Choose p ∈ [0,1] to maximize lb = $\max_{\mathrm{p}} \min(1 - \left(\frac{3}{2}\right)\mathrm{p},\ 2\mathrm{p} - 1)$

Maximized when $1 - \left(\frac{3}{2}\right)\mathrm{p} = 2\mathrm{p} - 1$, so p = 4/7, and lb = 1/7

What is the column player's minimax strategy?

Asymmetric Shooter-Goalie

	L	R
shooter L	$(-\frac{1}{2}, \frac{1}{2})$	$(1, -1)$
R	$(1, -1)$	$(-1, 1)$

Let **q** = $(q, 1 - q)$ be the minimax optimal goalie (column-player) strategy

Suppose shooter (row player) plays L

Goalie's payoff is $q \cdot \left(\frac{1}{2}\right) + (1 - q) \cdot (-1) = \frac{3q}{2} - 1$

Suppose shooter plays R

Goalie's payoff is $q \cdot (-1) + (1 - q) \cdot (1) = 1 - 2q$

Choose q ∈ [0,1] to realize $\max_q \min(\frac{3q}{2} - 1,\ 1 - 2q)$

$\frac{3q}{2} - 1 = 1 - 2q$ implies q = 4/7, and expected payoff at least -1/7

Remember: this means row player's ub at most 1/7

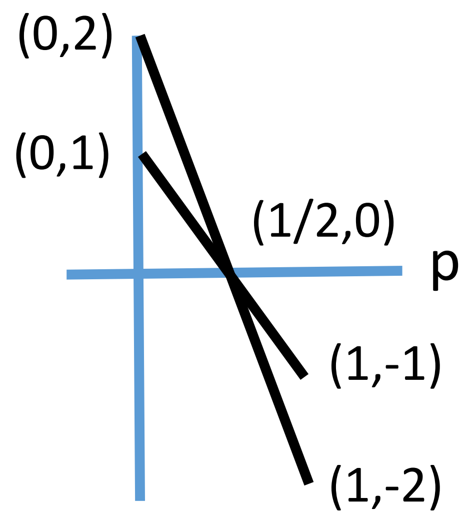
Uhh… lb = ub again… Value of the game is 1/7

Another Example

- Suppose in a zero-sum game, Row player's payoffs are:

-1	-2
1	2

- What is row player's minimax strategy? Why?
- Suppose her distribution is (p, 1-p)
- Expected payoff if column player plays first action is:
  $\mathrm{p} \cdot (-1) + (1 - \mathrm{p}) \cdot 1 = 1 - 2\mathrm{p}$
- Expected payoff if column player plays second action is:
  $\mathrm{p} \cdot (-2) + (1 - \mathrm{p}) \cdot 2 = 2 - 4\mathrm{p}$
- These lines both have a negative slope
- Should play p = 0
- Can show column player should always play first action and value of game is 1



Exercise 1: What if both players have somewhat different weaknesses? What if the payoffs are:

```
(-1/2, 1/2) (3/4, -3/4)
(1, -1)     (-3/2, 3/2)
```

Show that minimax-optimal strategies are $\mathbf{p} = (2/3, 1/3), \mathbf{q} = (3/5, 2/5)$ and value of game is 0.

Exercise 2: For the game with payoffs:

```
(-1/2, 1/2) (3/4, -3/4)
(1, -1)     (-2/3, 2/3)
```

Show that minimax-optimal strategies are $\mathbf{p} = (\frac{4}{7}, \frac{3}{7}), \mathbf{q} = (\frac{17}{35}, \frac{18}{35})$ and value of game is $\frac{1}{7}$.

Exercise 3: For the game with payoffs:

```
(-1/2, 1/2)    (-1, 1)
(1, -1)        (2/3, -2/3)
```

Show that minimax-optimal strategies are $\mathbf{p} = (0, 1), \mathbf{q} = (0, 1)$ and value of game is $\frac{2}{3}$.

Von Neumann's Minimax Theorem

- In each example,
  - row player has a strategy p^* guaranteeing a payoff of lb for her
  - column player has a strategy q^* guaranteeing row player's payoff is at most ub
  - lb = ub!

- Von Neumann: Given a finite 2-player zero-sum game,

$$\text{lb} = \max_{p} \min_{q} V_R(p, q) = \min_{q} \max_{p} V_R(p, q) = \text{ub}$$

  Common value is the *value of the game*

- In a zero-sum game, the row and column players can tell their strategy to each other and it doesn't affect their expected performance!
  - Don't tell each other your randomness

Lower Bounds for Randomized Algorithms

- A randomized algorithm is a zero-sum game

- Create a row-payoff matrix R:
  - Rows are possible inputs (for sorting, n!)
  - Columns are possible deterministic algorithms (e.g. every algorithm for sorting)
  - $R_{i,j}$ is cost of algorithm j on input i (e.g. number of comparisons)

- A deterministic algorithm with good worst-case guarantee is a column with entries that are all small

- A randomized algorithm with good expected guarantee is a distribution **q** on columns so the expected cost in each row is small

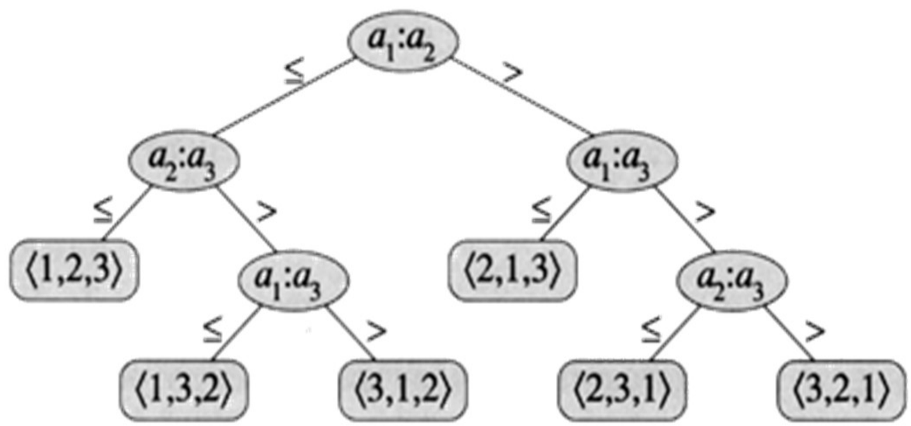
Lower Bounds for Randomized Algorithms

- Minimax-optimal strategy for column player is best randomized algorithm
- A lower bound for a randomized algorithm is a distribution $\mathbf{p}$ on inputs so for every algorithm j, expected cost of running j on input distribution **p** is large

  - $\max\limits_{\substack{\text{input} \\ \text{distributions p}}} \min\limits_{\substack{\text{deterministic} \\ \text{algorithms j}}} V_R(p, j) = \min\limits_{\substack{\text{randomized} \\ \text{algorithms q}}} \max\limits_{\text{inputs i}} V_R(i, q)$
  - show lb = $\max\limits_{\substack{\text{input} \\ \text{distributions p}}} \min\limits_{\substack{\text{deterministic} \\ \text{algorithms j}}} V_R(p, j)$ is large
  - give strategy for the row player (distribution on inputs) such that every column (deterministic algorithm) has high cost

Lower Bounds for Randomized Sorting

- Theorem: Let A be a randomized comparison-based sorting algorithm. There’s an input on which A makes an expected $\Omega(\lg n!)$ comparisons
- Proof: consider uniform distribution on n! permutations of n distinct numbers
- n! leaves
- No two inputs go to same leaf
- How many leaves at depth lg(n!) -10?
- $\leq$ 1+2+4+… + $2^{(\lg n!)-1} \leq \frac{n!}{512}$
- 511/512 > .99 fraction of inputs are at depth > lg(n!)-10
- Expected depth $> .99(\lg(n!) - 10) = \Omega(\lg n!)$



General-Sum Two-Player Games

- Many games are not zero-sum, have “win-win” or “lose-lose” payoffs
- Game of “chicken”
- Suppose two drivers facing each other each drive on their left (L) or right (R)

payoff matrix M	Bob L	Bob R
Alice L	$(1, 1)$	$(-1, -1)$
Alice R	$(-1, -1)$	$(1, 1)$

- What is a good notion of optimality to look at?

Nash Equilibria

- $(\mathbf{p}, \mathbf{q})$ is stable if no player has incentive to individually switch strategy
  - For any other strategy $\mathbf{p}'$ of row player,
    row player's new payoff $= \sum_{i,j} p_i' q_j R_{i,j} \leq \sum_{i,j} p_i q_j R_{i,j}$ = row player's old payoff
  - For any other strategy $\mathbf{q}'$ of column player,
    column player's new payoff $= \sum_{i,j} p_i \; q_j{}' C_{i,j} \leq \sum_{i,j} p_i q_j C_{i,j}$ = column player's old payoff

- For chicken, ((1,0),(1,0)) and ((0,1),(0,1)) and ((1/2,1/2),(1/2,1/2)) are Nash Equilibria

- Theorem (Nash): Every finite player game (with a finite number of strategies) has a Nash equilibrium