Lecture 11: Network Flow I

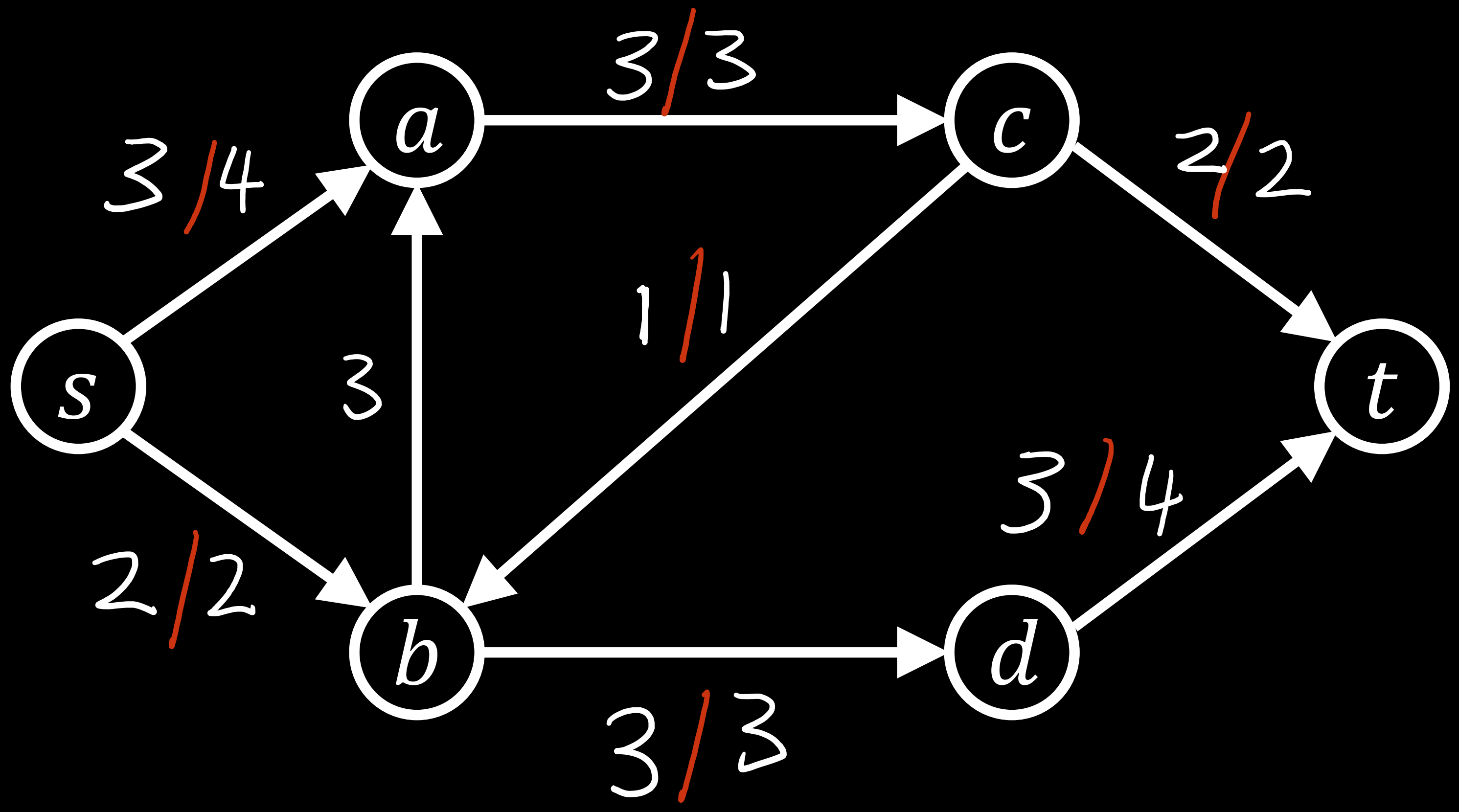
Flows, Cuts, and Matchings

Network Flow

Directed graph G

Edge capacity $c(u,v)$

Source s, sink t



Flow: $f: E \to \mathbb{R}$

Capacity: $0 \le f(u,v) \le c(u,v)$

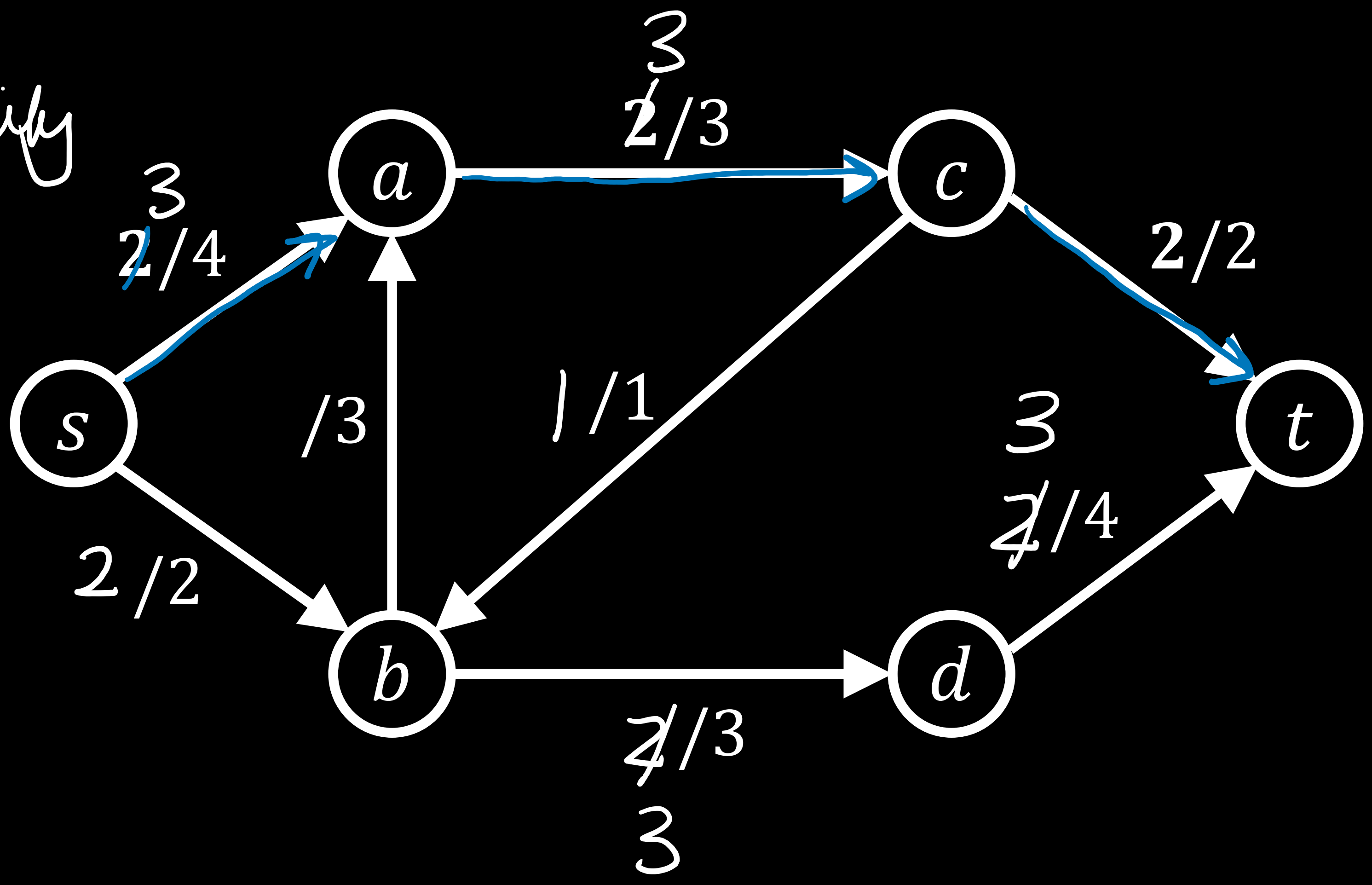
Flow conservation: "flow in = flow out" $\sum f(u,v) = \sum f(v,u)$

Goal: Maximum Flow !!

Improving a flow: *s*-*t* paths

If ∃ s-t path w/ capacity
then flow isn't max

Add flow to s-t path
w/o violating conservation



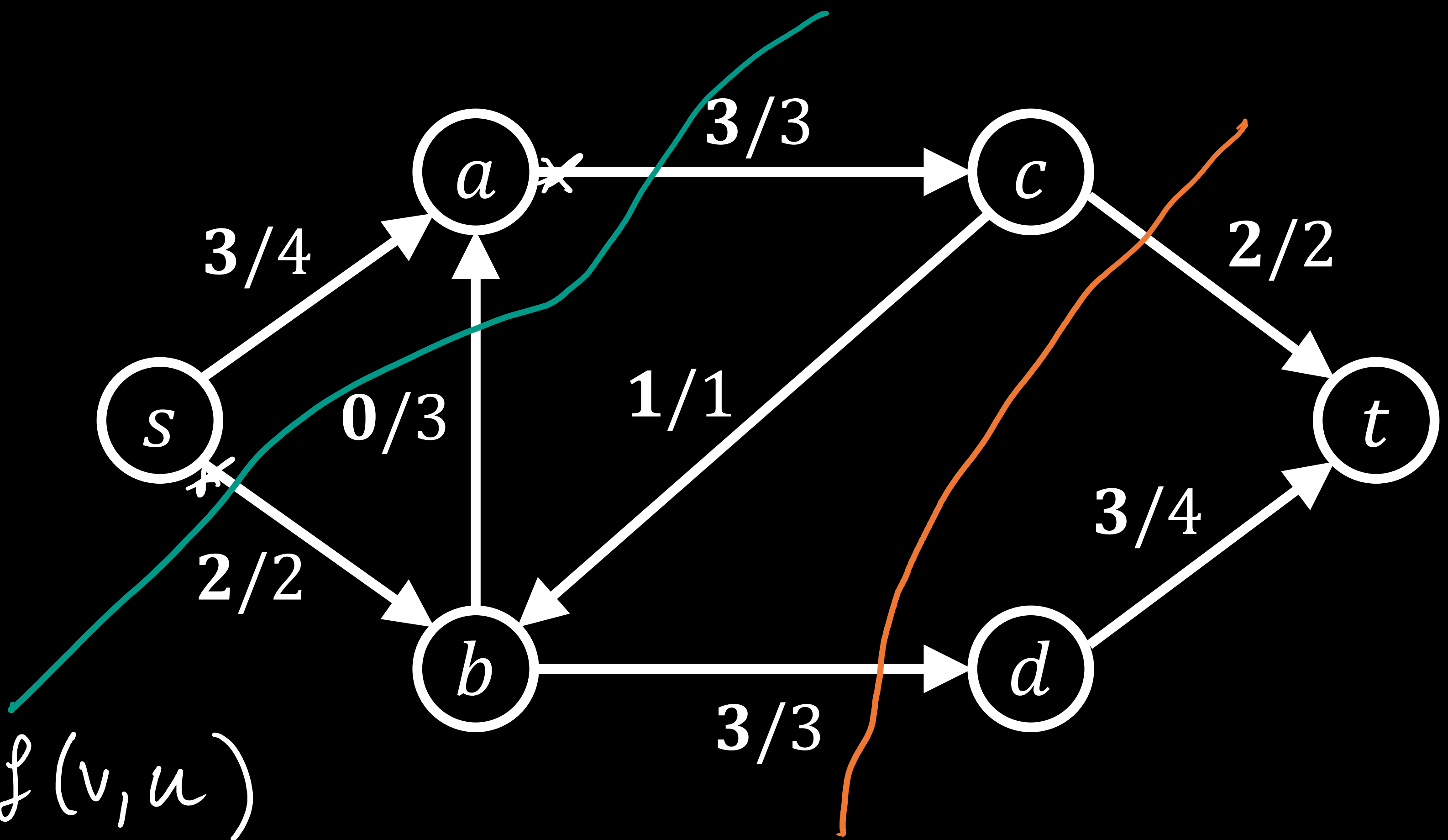
Certifying Optimality: s-t cuts

Cut: Separating into (S,T)

$s \in S \quad t \in T$

$$\mathrm{cap}(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v)$$

$$f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u)$$



any s-t flow $\le$ max flow $\le$ min cut $\le$ capacity any cut

Finding a maximum flow

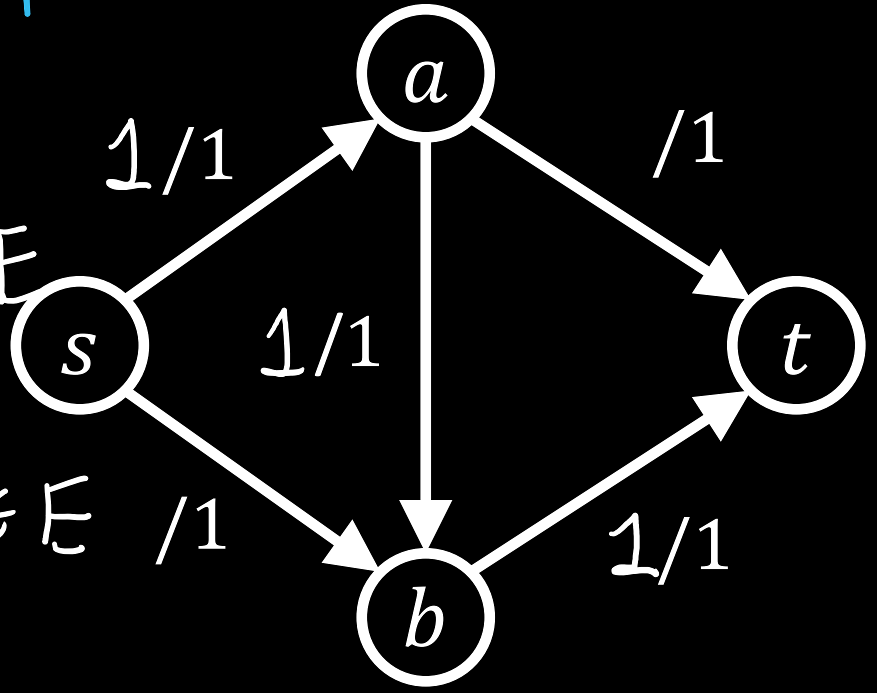
Local max ≠ Optimal

Residual graph

available cap

$$C_f(u,v) = \begin{cases} c(u,v) - f(u,v) & (u,v) \in E \\ f(v,u) & \text{if } (v,u) \in E \end{cases}$$

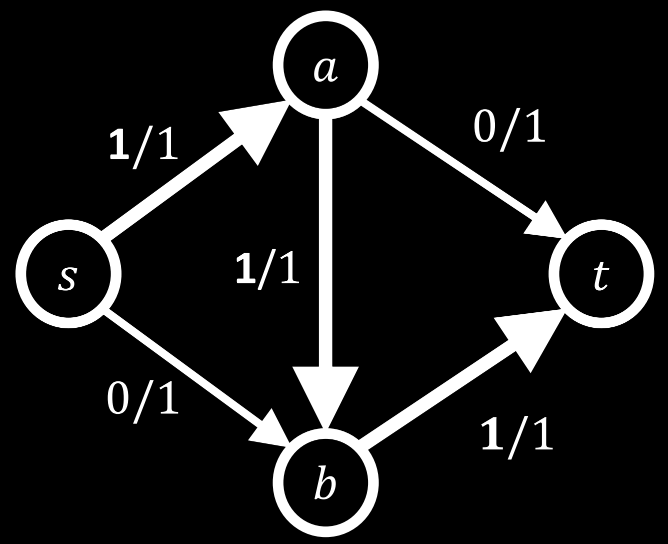
undo

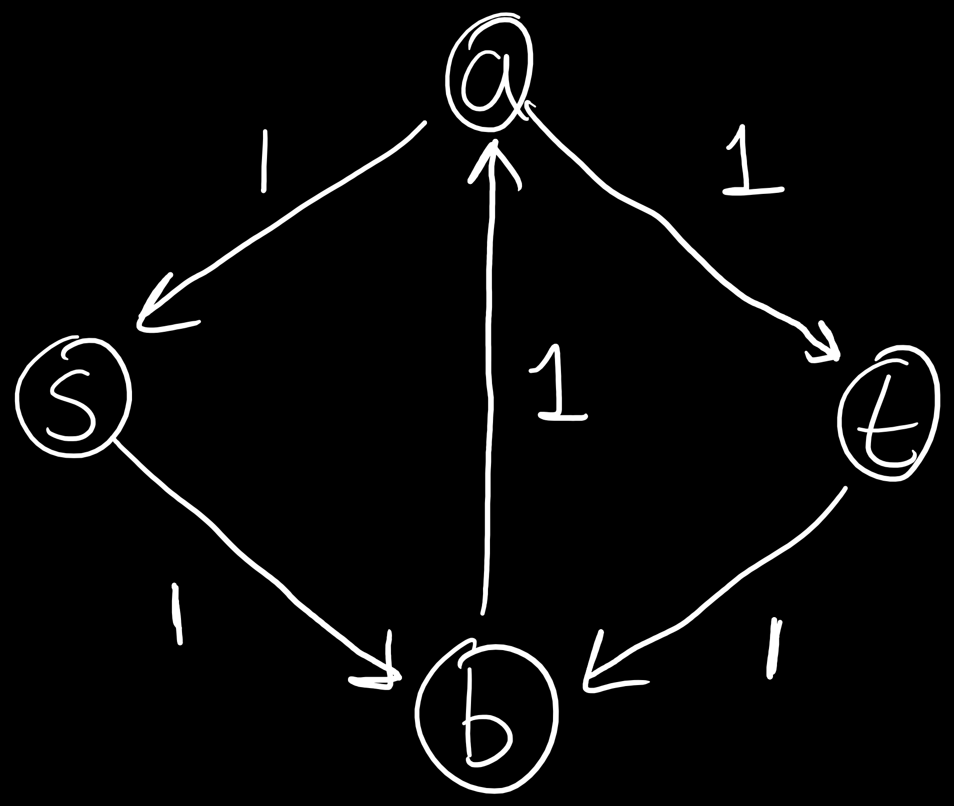


An s-t path in residual graph is called "augmenting path"

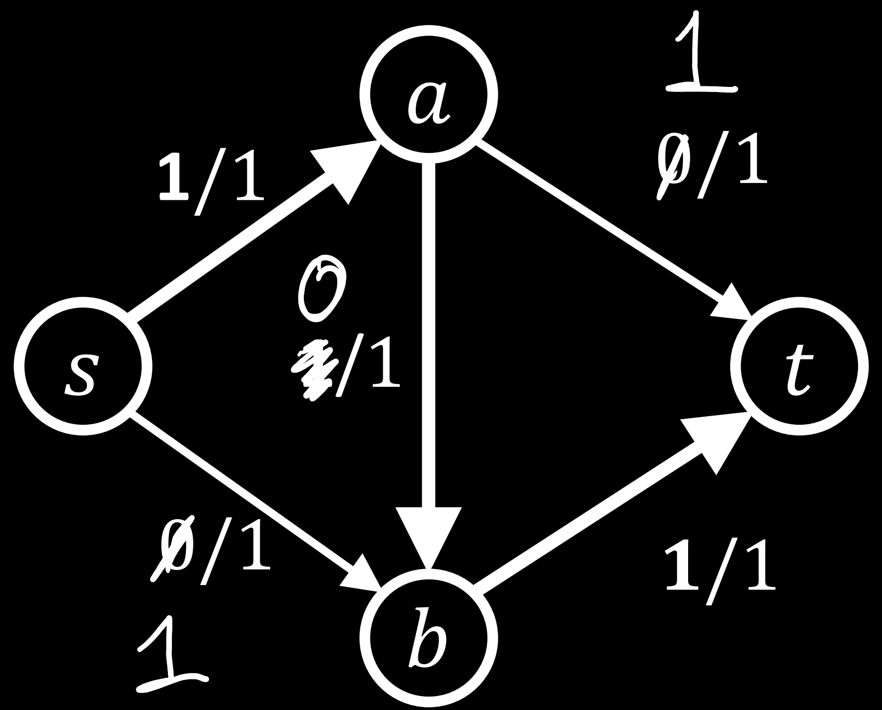
Need to “undo” bad decisions

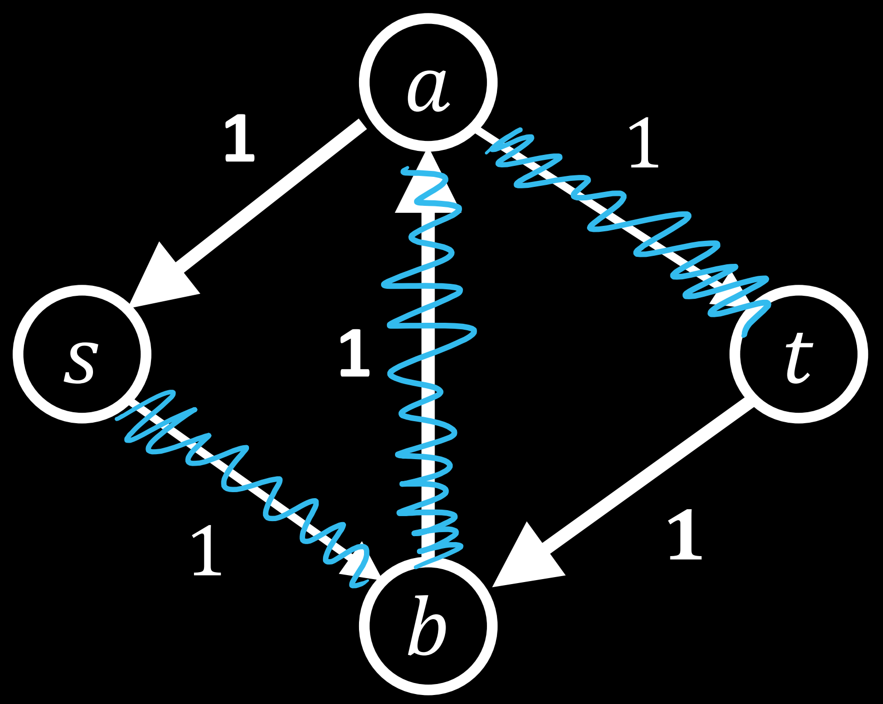
Ford-Fulkerson algorithm





Augmenting along a residual edge





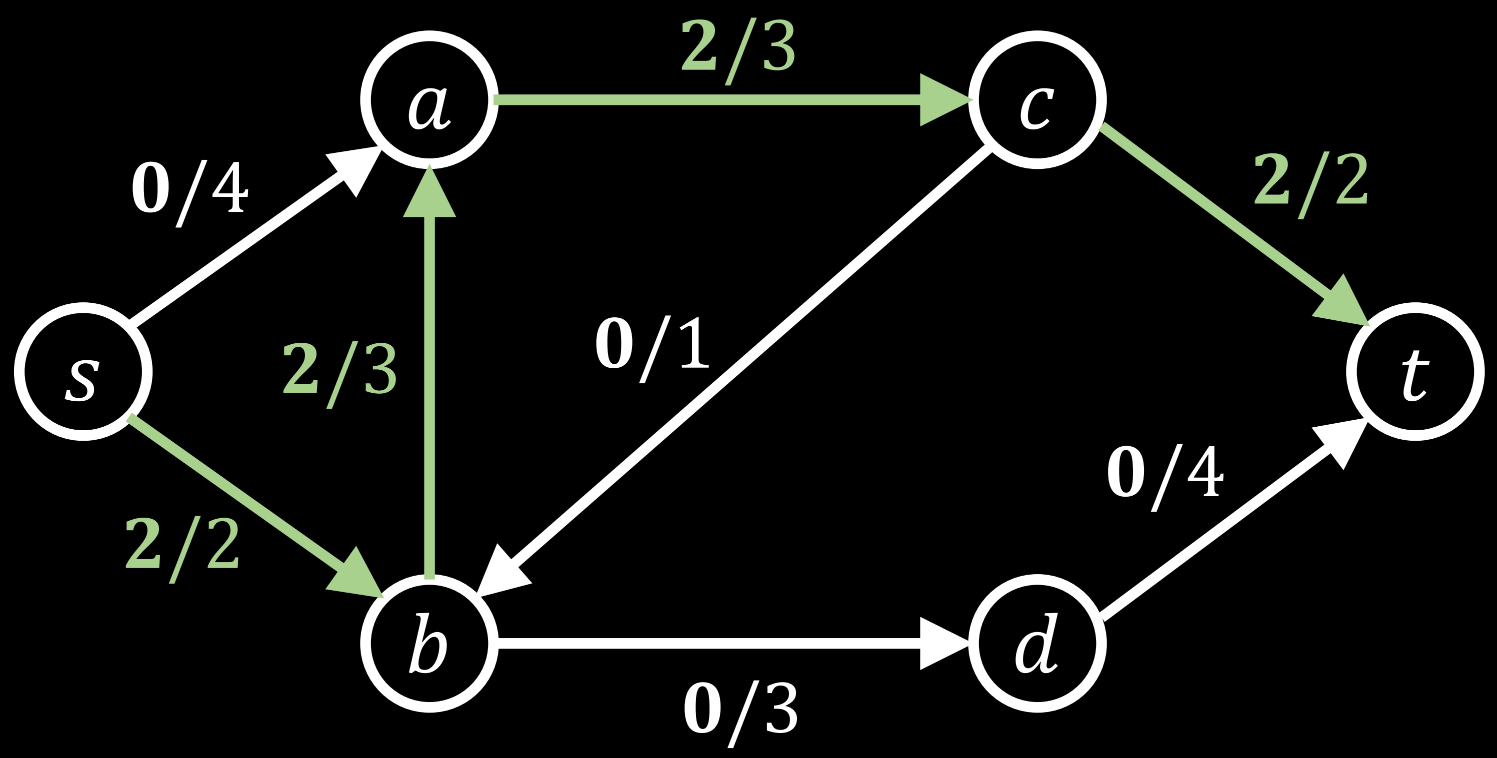
The Ford-Fulkerson Algorithm

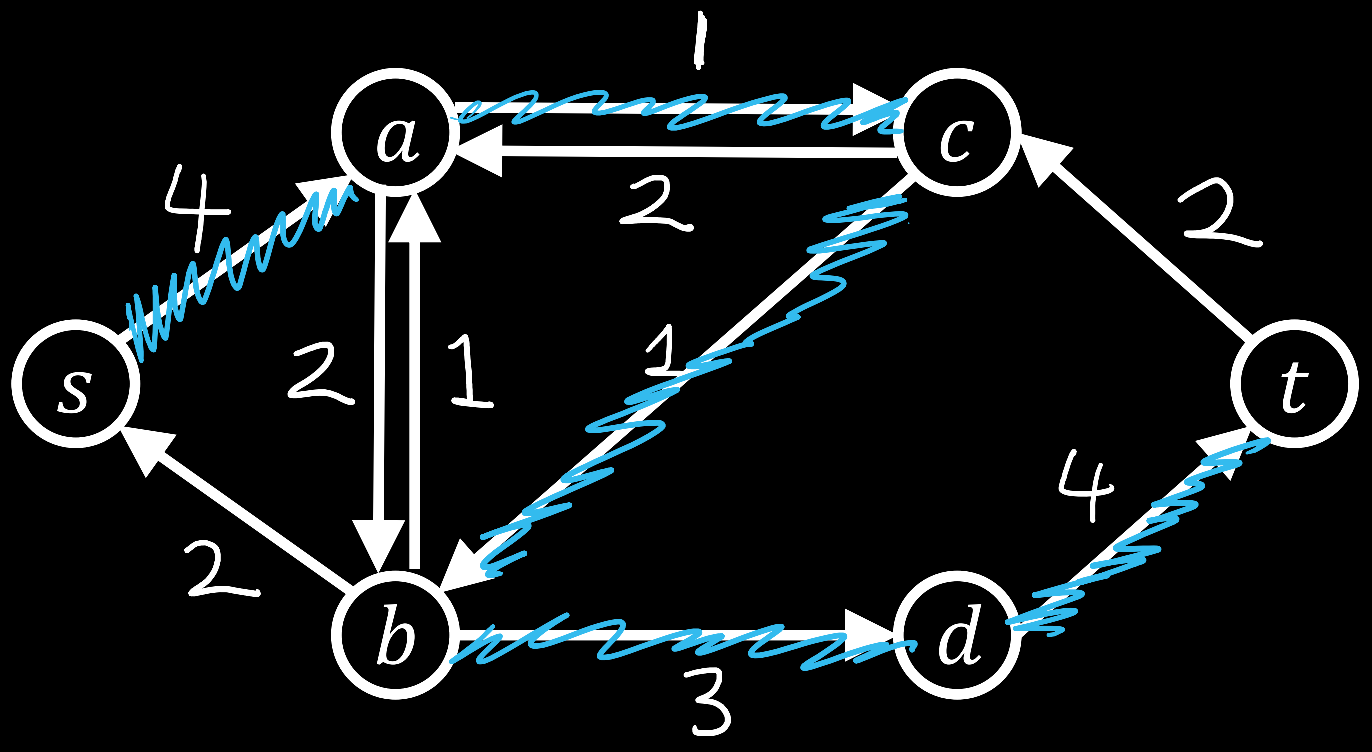
while exists an augmenting path
    push flow along it

(any augmenting path will do)

Ford-Fulkerson example

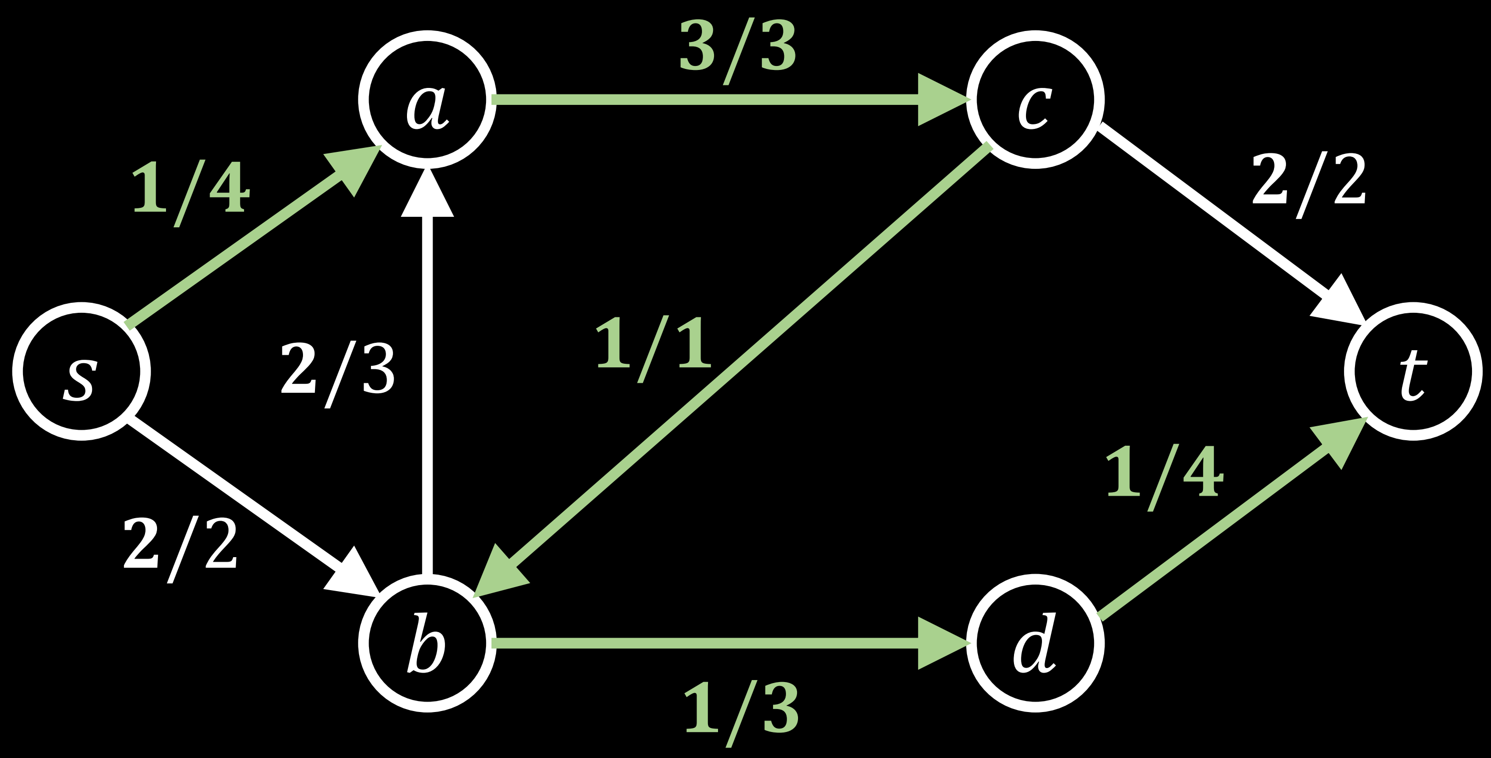
$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E, \\ f(v,u) & \text{if } (v,u) \in E. \end{cases}$$

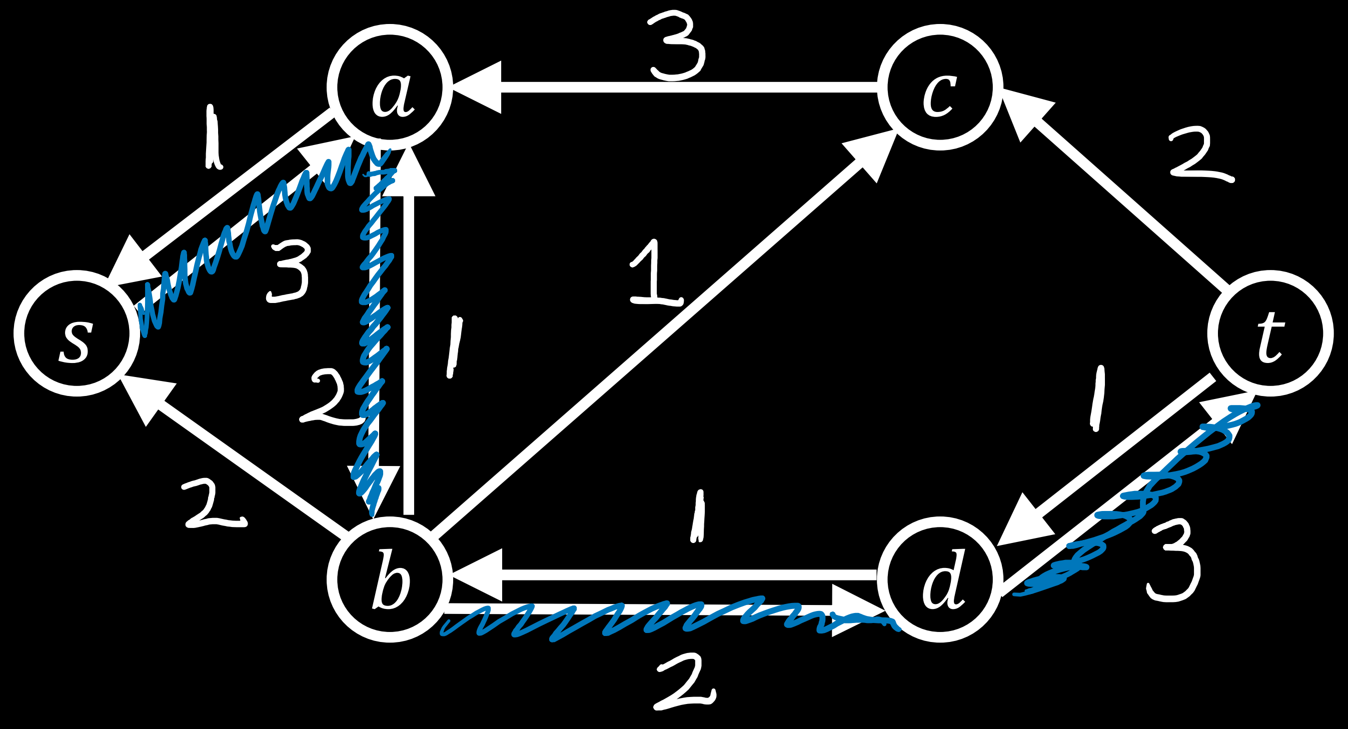




Ford-Fulkerson example

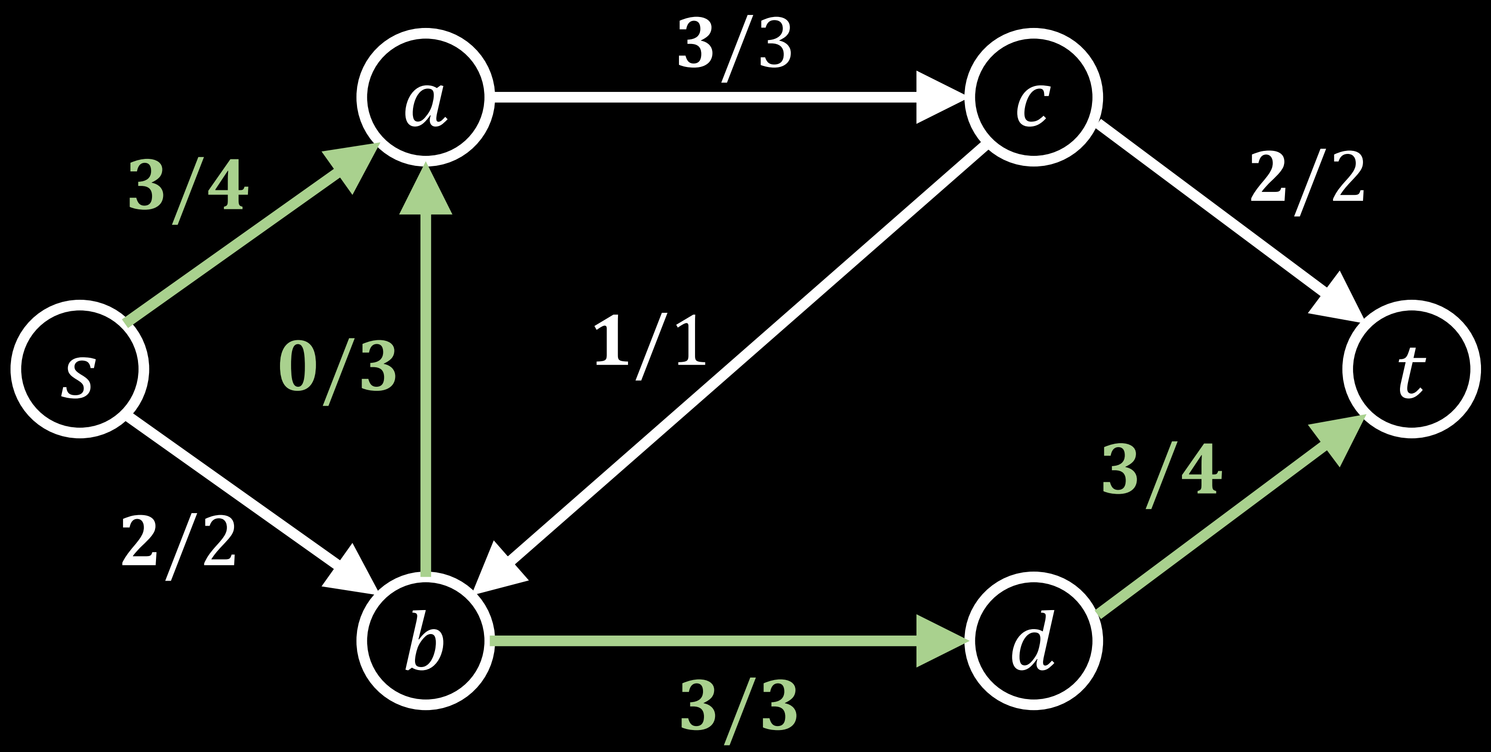
$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E, \\ f(v,u) & \text{if } (v,u) \in E. \end{cases}$$

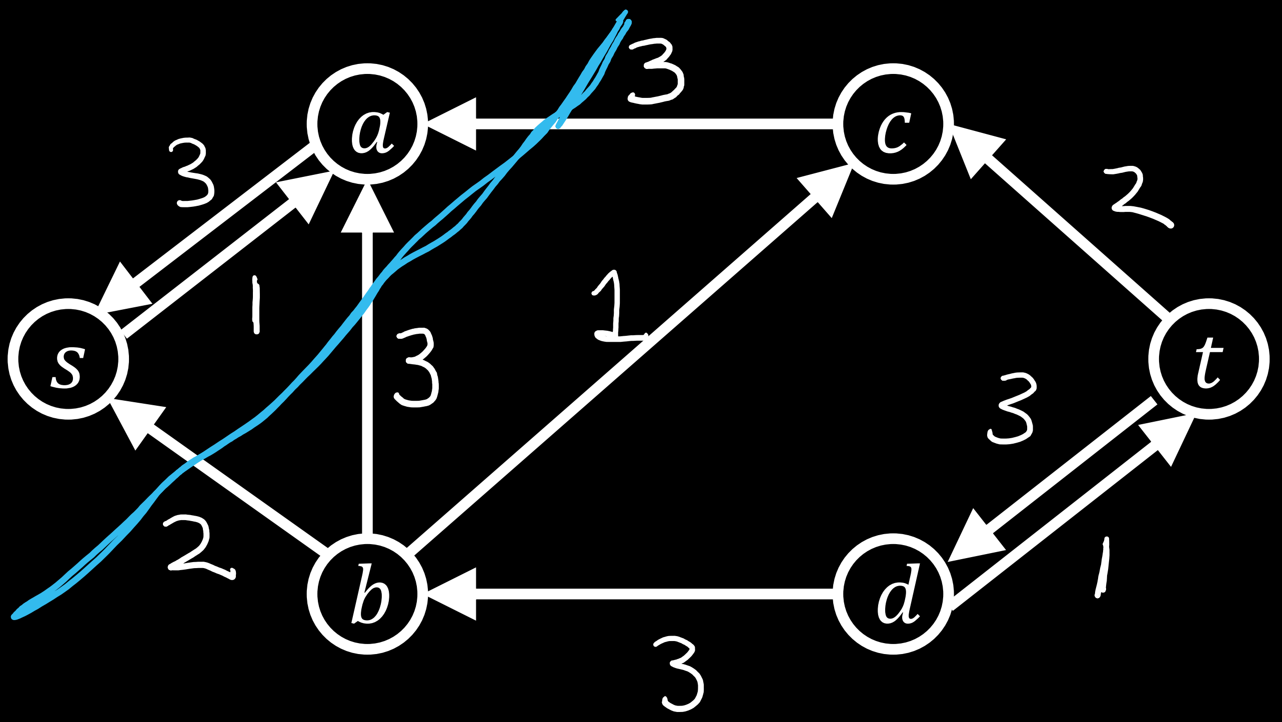




Ford-Fulkerson example

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E, \\ f(v,u) & \text{if } (v,u) \in E. \end{cases}$$



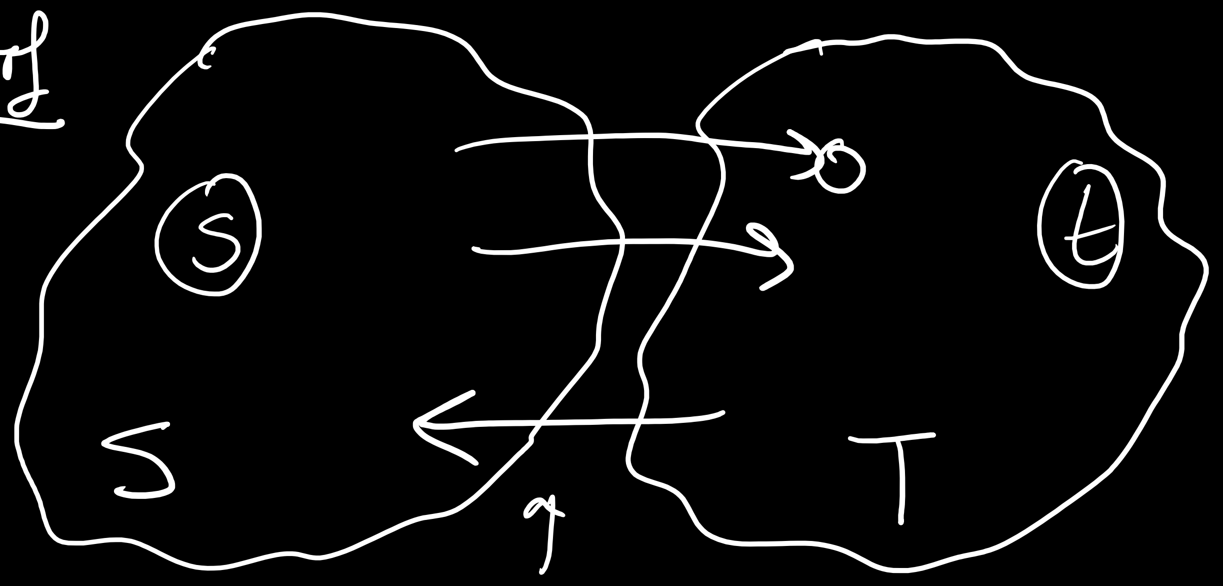


Analysis

Assumption: Capacities are integers

Theorem FF finds a flow with value = min cut

proof



Edges in G

S = stuff reachable from s in residual graph

Claim 1: Left to Right are full

Claim 2: Right to Left are empty

$$f(S,T) = cap(S,T)$$

$\square$

Implications

Theorem: (Min cut max flow theorem) Min cut = Max flow

Theorem: FF works

Theorem: (Integrality theorem) If capacities are integers, $\exists$ an integer max flow (every edge has integer flow)

Bonus How to find a min cut

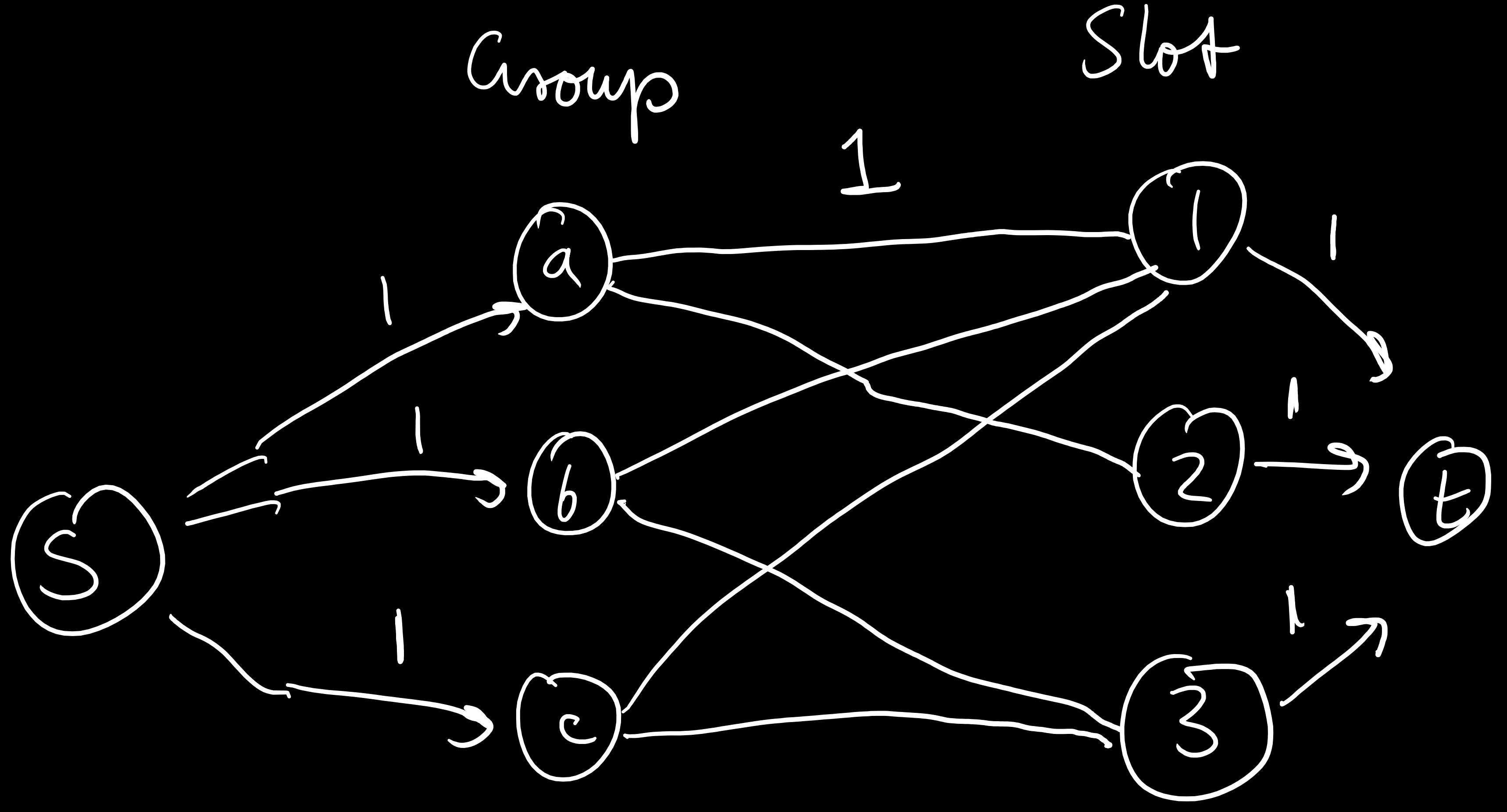
Running Time

Each iteration adds ≥ 1 flow

F iterations (F = max flow)

Runtime: $O(mF)$

Bipartite Matching



Matching is a set of edges with no endpoints in common

Flow ??

(Units of flow = matched edges.)

Analysis

- Didn’t finish the formal proof during the lecture. Please see the notes!