Lecture 6: The Data Stream Model

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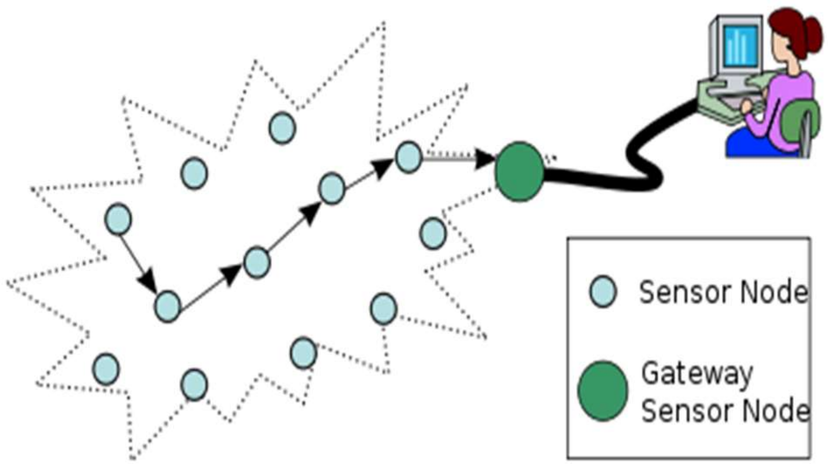
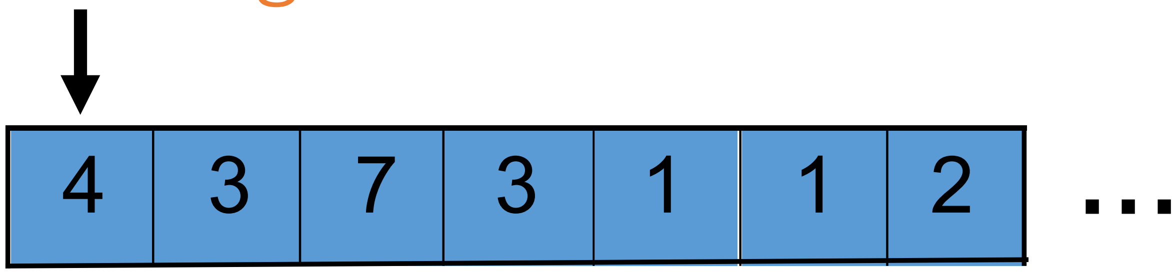
Data Streams

- A stream is a sequence of data, that is too large to be stored in available memory
- Examples
  - Internet search logs
  - Network Traffic
  - Sensor networks



  - Scientific data streams (astronomical, genomics, physical simulations)…

Streaming Model



- Stream of elements a_1, …, a_i, … each from an alphabet Σ and taking b bits to represent

- Single or small number of passes over the data

- Almost all algorithms are randomized and approximate
  - Usually necessary to achieve efficiency
  - Randomness is in the algorithm, not the input

- Goals: minimize space complexity (in bits), processing time

Example Streaming Problems

- Let $a_{[1:t]} =< a_1, \ldots, a_t >$ be the first t elements of the stream

- Suppose $a_1, \ldots, a_t$ are integers in $\{-2^b + 1, -2^b + 2, \ldots, -1, 0, 1, 2, \ldots, 2^b-1\}$
  - Example stream: 3, 1, 17, 4, -9, 32, 101, 3, -722, 3, 900, 4, 32

- How many bits do we need to maintain $f(a_{[1:t]}) = \sum_{i=1,\ldots,t} a_i$?
  - Outputs on example: 3, 4, 21, 25, 16, 48, 149, 152, -570, -567, 333, 337, 379, …
  - O(b + log t)

- How many bits do we need to maintain $f(a_{[1:t]}) = \max_{i=1,\ldots,t} a_i$?
  - Outputs on example: 3, 3, 17, 17, 17, 32, 101, 101, 101, 101, 900, 900, 900, …
  - O(b) bits

Example Streaming Problems

- The median of all the numbers we've stored so far
  - Example stream: 3, 1, 17, 4, -9, 32, 101, 3, -722, 3, 900, 4, 32
  - Median: 3, 1, 3, 3, 3, 3, 4, 3, …
  - This seems harder…

- The number of distinct elements we've seen so far?
  - Outputs on example: 1, 2, 3, 4, 5, 6, 7, 7, 8, 8, 9, 9, 9, …

- The elements that have appeared at least an ϵ-fraction of the time? These are the ϵ-heavy hitters
  - Cover today

Many Applications

- Internet router may want to figure out which IP connections are heavy hitters, e.g., the ones that use more than .01% of your bandwidth
- Or maybe the router wants to know the median (or 90-th percentile) of the file sizes being transferred
- Hashing is a key technique

Finding ϵ-Heavy Hitters

- S_t is the multiset of items at time t, so $S_0 = \emptyset, S_1 = \{a_1\}, \ldots, S_i = \{a_1, \ldots, a_i\}$,
$$\mathrm{count}_t(e) = |\{i \in \{1, 2, \ldots, t\} \text{ such that } a_i = e\}|$$

- $e \in \Sigma$ is an ϵ-heavy hitter at time t if $\mathrm{count}_t(e) > \epsilon \cdot t$

- Given $\epsilon > 0$, can we output the ϵ-heavy hitters?
  - Let's output a set of size $\frac{1}{\epsilon}$ containing all the ϵ-heavy hitters

- Note: can output "false positives" but not allowed to output "false negatives", i.e., not allowed to miss any heavy hitter, but could output non-heavy hitters

Finding ϵ-Heavy Hitters

- Example: E, D, B, D, D_5 D, B, A, C, B_{10} B, E, E, E, E_{15}, E
  (the subscripts are just to help you count)

- At time 5, the element D is the only 1/3-heavy hitter
- At time 11, both B and D are 1/3-heavy hitters
- At time 15, there is no 1/3-heavy hitter
- At time 16, only E is a 1/3-heavy hitter

Can't afford to keep counts of all items, so how to maintain a short summary to output the ϵ-heavy hitters?

Finding a Majority Element

- First find a .5-heavy hitter, that is, a majority element:

  memory $\leftarrow$ empty and counter $\leftarrow$ 0

  when element $\mathrm{a_t}$ arrives

  if (counter == 0)

  memory $\leftarrow \mathrm{a_t}$ and counter $\leftarrow 1$

  else

  if $\mathrm{a_t} =$ memory

  counter + +

  else

  counter - -

  (discard $\mathrm{a_t}$)
- At end of the stream, return the element in memory

$\cancel{3}$ $\cancel{1}$ $\cancel{2}$ $\cancel{1}$ 1

Memory = 3, Count = 1

Memory = 3, Count = 0

Memory = 2, Count = 1

Memory = 2, Count = 0

Memory = 1, Count = 1

Analysis of Finding a Majority Element

- If there is no majority element, we output a false positive, which is OK
- If there is a majority element, we will output it. Why?
- When we discard an element a_t, we throw away a different element
- When we throw away a copy of a majority element, we throw away another element
  - Either majority element is in memory, or majority element arrives in stream but some other item is in memory
- Majority element is more than half the total number of elements, so can't throw away all of them

Extending to ϵ-Heavy Hitters

Set $k = \left\lceil \frac{1}{\epsilon} \right\rceil - 1$

Array T[1, …, k], where each location can hold one element from Σ

Array C[1, …, k], where each location can hold a non-negative integer

C[i] ← 0 and $T[i]$ ←⊥ **for** all i

If there is j ∈ $\{1, 2, \ldots, k\}$ such that $a_t = T[j]$, **then** $C[j]++$

Else if some counter C[j] = 0 **then** $T[j] \leftarrow a_t$ and C[j] ← 1

Else decrement all counters by 1 (and discard element a_t)

$est_t(e) = C[j]$ if $e == T[j]$ for some j, and $est_t(e) = 0$ otherwise

Analyzing Counts

- Lemma: $0 \leq \mathrm{count}_t(e) - \mathrm{est}_t(e) \leq \frac{t}{k+1} \leq \epsilon \cdot t$

- Proof: $\mathrm{count}_t(e) \geq \mathrm{est}_t(e)$ since we never increase a counter for e unless we see e

If we don't increase $\mathrm{est}_t(e)$ by 1 when we see an update to e, we decrement k counters and discard the current update to e

So we drop k+1 distinct stream updates, but there are t total updates, so we won't increase $\mathrm{est}_t(e)$ by 1, when we should, at most $\frac{t}{k+1} \leq \epsilon \cdot t$ times

Heavy Hitters Guarantee

- At any time t, all ϵ-heavy hitters e are in the array T. Why?
- For an ϵ-heavy hitter e, we have $\mathrm{count}_t(\mathrm{e}) > \epsilon \cdot \mathrm{t}$
- But $\mathrm{est}_t(\mathrm{e}) \geq \mathrm{count}_t(\mathrm{e}) - \epsilon \cdot \mathrm{t}$
- So $\mathrm{est}_t(\mathrm{e}) > 0$, so e is in array T
- Space is $O(k (\log(\Sigma) + \log t)) = O(1/\epsilon) (\log(\Sigma) + \log t)$ bits

Heavy Hitters with Deletions

- Suppose we can delete elements e that have already appeared

- Example: (add, A), (add, B), (add, A), (del, B), (del, A), (add, C)

- Multisets at different times
  $\mathrm{S}_0 = \emptyset, \mathrm{S}_1 = \{\mathrm{A}\}, \mathrm{S}_2 = \{\mathrm{A}, \mathrm{B}\}, \mathrm{S}_3 = \{\mathrm{A}, \mathrm{A}, \mathrm{B}\}, \mathrm{S}_4 = \{\mathrm{A}, \mathrm{A}\}, \mathrm{S}_5 = \{\mathrm{A}\},$
  $\mathrm{S}_6 = \{\mathrm{A}, \mathrm{C}\}, \ldots$

- "active" set S_t has size $|\mathrm{S}_\mathrm{t}| = \sum_{\mathrm{e} \in \Sigma} \mathrm{count}_\mathrm{t}(\mathrm{e})$ and can grow and shrink

Data Structure for Approximate Counts

- Query "What is $\mathrm{count}_t(e)$?", should output $\mathrm{est}_t(e)$ with:
$$\Pr[|\mathrm{est}_t(e) - \mathrm{count}_t(e)| \leq \epsilon |S_t|] \geq 1 - \delta$$
- Want space close to our previous O(1/ ϵ) (log(Σ) + log t) bits
- Let $h: \Sigma \to \{0,1,2, \ldots, k-1\}$ be a hash function (will specify later)
- Maintain an array A[0, 1, …, k-1] to store non-negative integers

when update a_t arrives:

if $a_t = (\mathrm{add}, e)$ **then** $A[h(e)] + +$

else $a_t = (\mathrm{del}, e)$, and $A[h(e)] - -$

- $\mathrm{est}_t(e) = A[h(e)]$

Data Structure for Approximate Counts

- $\mathrm{A}[\mathrm{h}(\mathrm{e})] = \sum_{\mathrm{e}' \in \Sigma} \mathrm{count}_\mathrm{t}(\mathrm{e}') \cdot \mathbf{1}(\mathrm{h}(\mathrm{e}') = \mathrm{h}(\mathrm{e}))$, where **1**(condition) evaluates to 1 if the condition is true, and evaluates to 0 otherwise
- $\mathrm{A}[\mathrm{h}(\mathrm{e})]$ = $\mathrm{count}_\mathrm{t}(\mathrm{e}) + \sum_{\mathrm{e}' \neq \mathrm{e}} \mathrm{count}_\mathrm{t}(\mathrm{e}') \cdot \mathbf{1}(\mathrm{h}(\mathrm{e}') = \mathrm{h}(\mathrm{e}))$,
- $\mathrm{est}_\mathrm{t}(\mathrm{e}) - \mathrm{count}_\mathrm{t}(\mathrm{e}) = \sum_{\mathrm{e}' \neq \mathrm{e}} \mathrm{count}_\mathrm{t}(\mathrm{e}') \cdot \mathbf{1}(\mathrm{h}(\mathrm{e}') = \mathrm{h}(\mathrm{e}))$
- Since we have a small array A with k locations, there are likely many $\mathrm{e}' \neq \mathrm{e}$ with h(e') = h(e), but can we bound the expected error?

Data Structure for Approximate Counts

- Recall: Family H of hash functions h: U -> {0, 1, …, k-1} is universal if for all $x \neq y$,
$$\Pr_{h \leftarrow H}[h(x) = h(y)] \leq \frac{1}{k}$$
- Gave a simple family where h can be specified using O(log |U|) bits. Here, |U| = |Σ|

- $E[\mathrm{est}_t(e) - \mathrm{count}_t(e)] = E[\sum_{e' \neq e} \mathrm{count}_t(e') \cdot \mathbf{1}(h(e') = h(e))]$
$$= \sum_{e' \neq e} \mathrm{count}_t(e') \cdot E[\,\mathbf{1}(h(e') = h(e))]$$
$$= \sum_{e' \neq e} \mathrm{count}_t(e') \cdot \Pr[h(e') = h(e)]$$
$$\leq \sum_{e' \neq e} \mathrm{count}_t(e') \cdot \left(\frac{1}{k}\right)$$
$$= \frac{|S_t| - \mathrm{count}_t(e)}{k} \leq \frac{|S_t|}{k}$$

k = $1/\epsilon$ makes this at most $\epsilon \cdot |S_t|$. Space is $O(\frac{1}{\epsilon})$ counters plus storing hash function

High Probability Bounds for CountMin

- Have $0 \leq \mathrm{est}_t(e) - \mathrm{count}_t(e) \leq |S_t|/k$ in expectation from CountMin
  - With probability 1/2, $\mathrm{est}_t(e) - \mathrm{count}_t(e) \leq 2|S_t|/k$ Why?

- Can we make the success probability 1-δ?
  - Independent repetition: pick m hash functions $h_1, \ldots, h_m$ with $h_i: \Sigma \rightarrow \{0, 1, 2, \ldots, k-1\}$ independently from H. Create array A_i for h_i

when update a_t arrives:

for each i from 1 to m

if $a_t = (\mathrm{add}, e)$ **then** $A_i[h_i(e)]++$

else $a_t = (\mathrm{del}, e)$ and $A_i[h_i(e)]--$

High Probability Bounds and Overall Space

What is our new estimate of $\mathrm{count}_t(\mathrm{e})$?

$$\mathtt{best}_t(e) := \min_{i=1}^{m} A_i[h_i(e)].$$

- Each $\mathrm{A_i[h_i(e)]}$ is an *overestimate* to $\mathrm{count_t(e)}$
- By independence, Pr[for all i, $\mathrm{A_i[h_i(e)]} - \mathrm{count_t(e)} \geq 2|\mathrm{S_t}|/\mathrm{k}] \leq \left(\frac{1}{2}\right)^m$
- For $\mathrm{k} = \frac{2}{\epsilon}$ and m = $\log_2\left(\frac{1}{\delta}\right)$, the error is at most $\epsilon|\mathrm{S_t}|$ with probability 1-δ
- Space: $\mathrm{m} \cdot \mathrm{k} = \mathrm{O}(\frac{\log\left(\frac{1}{\delta}\right)}{\epsilon})$ counters each of O(lg t) bits

  $\mathrm{m} \cdot \mathrm{O}(\log|\Sigma|) = \mathrm{O}(\log\left(\frac{1}{\delta}\right)\log|\Sigma|)$ bits to store hash functions

ϵ-Heavy Hitters

- Our new estimate $\mathrm{best}_t(e)$ satisfies
$$\Pr[|\mathrm{best}_t(e) - \mathrm{count}_t(e)| \le \epsilon |S_t|] \ge 1 - \delta$$

  and uses $O(\frac{\log\left(\frac{1}{\delta}\right) \log t}{\epsilon} + \log\left(\frac{1}{\delta}\right) \log|\Sigma|)$ bits of space

- What if we want with probability 9/10, simultaneously for all e, $|\mathrm{best}_t(e) - \mathrm{count}_t(e)| \le \epsilon |S_t|$?

- Set $\delta = \frac{1}{10|\Sigma|}$ and apply a union bound over all $e \in \Sigma$