1 Preliminaries

We'll give two algorithms for the following *closest pair* problem:

> Given n points in the plane, find the pair of points that is the closest together.

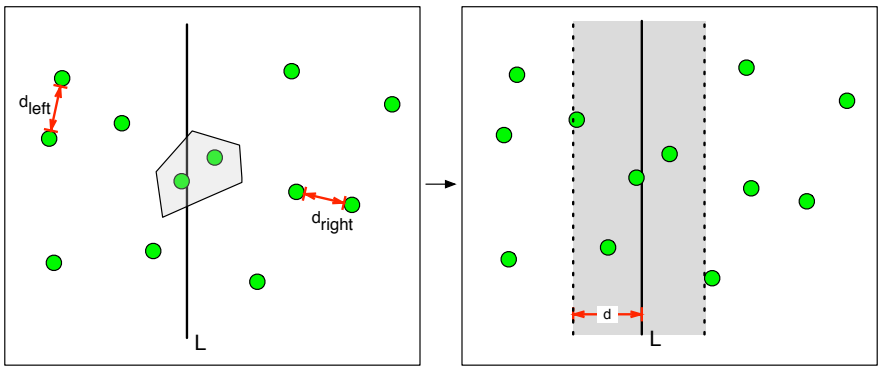
The first algorithm is a deterministic divide and conquer and runs in $O(n \log n)$. The second one is random incremental and runs in expected time $O(n)$. In this lecture we make the following assumptions:

> We assume the points are presented as real number pairs (x, y).
> We assume arithmetic on reals is accurate and runs in $O(1)$ time.
> We will assume that we can take the floor function of a real.
> We also assume that hashing is $O(1)$ time.

These assumptions (in this context) are resonable, because the algorithms will not abuse this power.

2 $O(n \log n)$ Divide and Conquer Algorithm

Clearly, we can solve the problem in $O(n^2)$ time, but in fact we can do better. The main idea is to divide the points in half, and recursively find the closest pair of points in each half. The tricky part will be the case when the closest pair of points spans the line that divides the points in half, like the shaded pair below:



To handle such pairs, we look at the "slab" of width 2δ centered on the dividing line L. For each of the points there, we check its distance against other points in the slab. The key fact is that for each point in the slab, we will need to check it against only a few points. We see why soon, but first, here is the pseudocode for the algorithm:

ClosestPair $(p_1, p_2, \ldots, p_n)$:

// The points are in sorted order by x.
// We also have the points in a different list ordered by y

If $n \leq 3$ then solve and return the answer.
Let $m = \lfloor n/2 \rfloor$
Let $\delta = \min(\text{ClosestPair}(p_1, \ldots, p_m), \text{ClosestPair}(p_{m+1}, \ldots, p_n))$

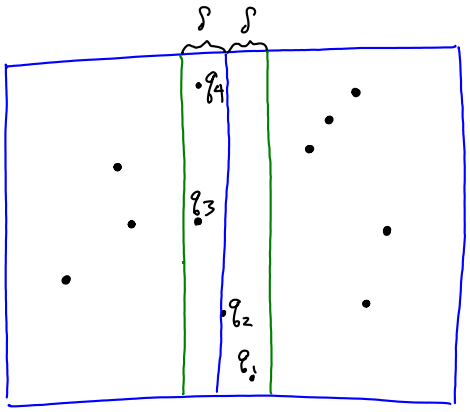
Form a list q of the points (sorted by increasing y) that are within δ of the x coordinate of p_m. Call these points $q_1, q_2 \ldots q_k$

Now we compute d_i, the minimum distance between q_i and all the qs below it, for each $1 \leq i \leq k$ as follows:

$d_i =$ min distance from q_i to $q_{i-1}, q_{i-2}, \ldots, q_j$, where we stop when j gets to 1, or the y coordinate gets too small: $q_j \cdot (0,1) < q_i \cdot (0,1) - \delta$

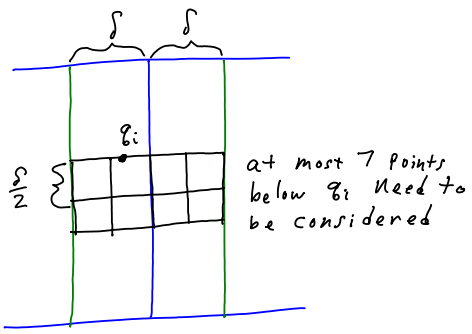
Return $\min(d_1, \ldots, d_k, \delta)$

The divide and conquer approach is obvious. The closest pair is either within the left half, within the right half, or it has one endpoint in the left half and one in the right half. If it is that last case, it's clear that both of the points must be within δ of the dividing line, so it is that middle region that the merge step focuses on:



We need to determine if there is a pair that straddles the dividing line that has a distance less than δ. This is accomplished by the inner loop that computes d_i. It's obviously correct, because any pairs that the loop does not consider are too far apart. (Either they are not in the 2δ-wide slab, or their y coordinates differ by at least δ.) The only tricky part is proving that the inner loop (where we compute the distance from q_i to $q_{i-1}, \ldots, q_j$) finds a stopping value of j in $O(1)$ time.

The key idea is to divide the slab into a grid of the appropriate size for the analysis. We use a grid of squares that are $\delta/2$ on a side:



In this figure there is a four (across) by two (down) grid of squares of size $\delta/2$. The top of the grid goes through q_i. Note that each of these squares can contain at most one of the points (interior or on its boundary), because any two points in the square are at most $\delta/\sqrt{2} < \delta$ apart. Therefore any q that is below the bottom of this grid is at least δ away from q_i, and thus has no effect on the closest pair distance. This proves that there are at most 7 points besides below q_i that are tested before the loop terminates. So for each i this search is $O(1)$ time.

The initiation phase sorts by x and also by y. This is $O(n \log n)$. Subsequent phases just filter these sorted lists, and no further sorting is done.

So the algorithm partitions the points ($O(n)$), does two recursive calls of $n/2$ in size, scans the points again to form the list q ($O(n)$), then scans the list q looking for the closest side-crossing pair ($O(n)$).

So we get the classic divide and conquer recurrence:

$$T(n) = 2T(n/2) + n$$

which solves to $O(n \log n)$.

3 Sariel Har-Peled's Randomized $O(n)$ Algorithm for closest pair

For any set of points P, let $\mathrm{CP}(P)$ be the closest pair distance in P.

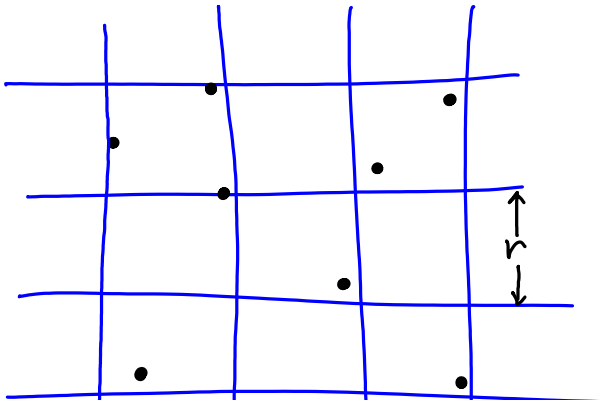
We're going to define a "grid" data structure, and an API for it. The grid (denoted G) stores a set of points (that we'll call P) and also stores the closest pair distance r for those points. The number r is called "the grid size of G". In other words, we maintain the invariant that the grid size will always be equal to the closest distance between any points currently in the grid. Here's the API:

MakeGrid(p, q): Make and return a new grid containing points p and q using $r = |p - q|$ (the distance between p and q) as the initial grid size.

Lookup(G, p): p is a point, G is a grid. This returns two types of answers. Let r' be the closest distance from p to a point in P. If $r' < r$ then return r'. If $r' \geq r$ return "Not Closest". Note that Lookup() does not need to compute r' if $r' \geq r$.

Insert(G, p): G is a grid. p is a new point not in the grid. This inserts p into the grid. It returns the new grid size.

Here's an example Grid data structure:

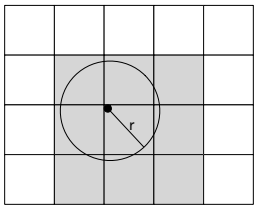


We now discuss how to implement this API. First define a function Boxify$((x, y), r)$. This returns the integer point $(\lfloor x/r \rfloor, \lfloor y/r \rfloor)$. This gives the grid "box" that a point belongs to.

The data structure maintains a hash table whose keys are these integer pairs. The values in the hash table are lists of points from P. So the key (i, j) (also called a *box*) in the hash table stores all the points of P whose Boxify() value is (i, j).

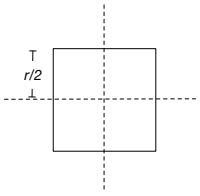
MakeGrid(p, q) is trivial. Just insert p and q into a new table with $r = |p - q|$.

Lookup(G, p) computes Boxify(p, r). It then looks in that box, and the 8 surrounding ones, and computes the distance between p and the closest one of these. Call this number r'. If $r' < r$, then we return r'. If $r' > r$ then return "Not Closest". This works because we know that if there is a point closer to p than r, it must be in one of the 9 boxes that are searched by this function:



For example, in the grid with points at the top of this page, suppose a new point p lands in the middle square of this grid. Now to determine if it is closer to another point than the grid size, all we have to do is examine the points in each of the nine boxes shown. (In this case it is eight points.)

Also note that the running time of this is $O(1)$ because it does 9 lookups in the hash table, and the total number of points it has to consider is at most 36. This is because a box contains at most 4 points by a similar argument as we made for the previous closest pair algorithm:



Each of the $r/2$ subregions can have at most one point in it.

Insert(G, p) works as follows. It first does a Lookup(G, p). If the result is "Not Closest" it just inserts p into the data structure into box Boxify(p, r). This is correct, since it means that p does not create a new closest pair, so the grid size should be unchanged. This is $O(1)$ time. On the other hand if the Lookup() returns $r' < r$, then the algorithm rehashes every point into a new hash table based on the new grid size being r'. This takes $O(i)$ time if there are i points now being stored in the data structure.

These algorithms are clearly correct, simply by virtue of the fact that at any point in time the grid size r is equal to the CP(P) where P is the set of points in the data structure. This is preserved by all the operations.

We can now complete the description of the algorithm:

Randomized-CP(P):
    Randomly permute the points. Call the new ordering $p_1, p_2, \ldots, p_n$.
    $G =$ MakeGrid(p_1, p_2)
    for $i = 3$ to n do
        $r =$ Insert(G, p_i)
    done
    return r

Claim: This algorithm computes CP(P).

Proof: Follows from the definition of the API. ■

Claim: The algorithm runs in expected $O(n)$ time.

Proof: Recall the time to do Insert() is $O(1)$ if the grid size does not change, and $O(i)$ ($i =$ the number of points in the grid) if the grid size does change.

We use an argument similar to the one we used for Seidel's LP algorithm, called "backward analysis":

Consider running the algorithm backwards. Here we are deleting points in order $p_n, p_{n-1}, \ldots, p_3$. When deleting point i, the operation is $O(1)$ if the closest pair distance does not change, and $O(i)$ if it does. In general if you remove a random point from a set of i points, the probability that the closest pair distance changes is at most $2/i$: if there is just one pair of points that achieves the minimum distance, then you have to remove one of those two points to increase the minimum. If there is more than one pair, then the probability is lower.

So the removal is costly (i.e. $O(i)$) with probability at most $2/i$, and cheap $O(1)$ with the remaining probability. This is exactly the same situation as in Seidel's LP algorithm. Therefore the expected cost of a step is $O(1)$. Thus the expected cost of the entire algorithm is $O(n)$. ■

On the next page you can find code implementing this algorithm in Ocaml.

```
(*  Sariel Har-Peled's linear time algorithm for closest pairs.
    Danny Sleator, Nov 2014
*)

let sq x = x *. x

(* The function below takes an array of points (float*float), and returns
   the distance between the closest pair of points.  *)
let closest_pair p =
  let n = Array.length p in

  let dist i j =
    let (xi,yi) = p.(i) in
    let (xj,yj) = p.(j) in
    sqrt ((sq (xi -. xj)) +. (sq (yi -. yj)))
  in

  let truncate x r = int_of_float (floor (x /. r)) in
  let boxify (x,y) r = (truncate x r, truncate y r) in

  let getbox h box = try Hashtbl.find h box with Not_found -> [] in

  let add_to_h h i box =
    Hashtbl.replace h box (i::(getbox h box))
  in

  let make_grid i r =
    (* put points p.(0) ... p.(i) into a new grid of size r.
       it has already been established that the closest pair
       in that point set has distance r *)
    let h = Hashtbl.create 10 in

    for j=0 to i do
      add_to_h h j (boxify p.(j) r)
    done;
    h
  in

Random.self_init ();
let swap i j =
  let (pi,pj) = (p.(i),p.(j)) in
  p.(i) <- pj;  p.(j) <- pi
in
for i=0 to n-2 do
  let r = Random.int (n-i) in
  swap i (i+r)
done;

let rec loop h i r =
  (* already built the table for points 0...i, and they have dist r *)

  if i=n-1 then r else
    let i = i+1 in
    let (ix,iy) = boxify p.(i) r in
    let li = ref [] in
    for x = ix-1 to ix+1 do
      for y = iy-1 to iy+1 do
        li := (getbox h (x,y)) @ !li
      done
    done;

    let r' = List.fold_left (
      fun ac j -> min (dist i j) ac
    ) max_float !li in

    if r' < r then (
      loop (make_grid i r') i r'
    ) else (
      add_to_h h i (ix,iy);
      loop h i r
    )
in

let r0 = dist 0 1 in
loop (make_grid 1 r0) 1 r0
```