451/651 Lecture 19 – Geometric Algorithms

Solving low-dimensional problems involving geometric objects ...
points, lines, circles, polygons, etc.

In addition to their practical applications, these problems and their algorithms lead to a lot of very interesting and generally applicable data structures and algorithmic techniques.

Part 1: 2-Dimensional Geometric Primitives.

- A point: It's generally represented as a pair p = (x,y) of real numbers. (Other representations are possible.)
- Real Arithmetic Assumption: We allow arithmetic on real numbers and assume that it takes O(1) time.
- This assumption is not to be abused.

Primitives, contd

Adding and subtracting points:

Cross Product:

Dot Product:

Scalar Mult:

Length:

Normalization:

Rotation:

Meaning of the cross product

Meaning of the dot product

Of course we've seen this before. The objective function of an LP is a dot product.

The line through $(0, 0)$ perpendicular to the line $((0, 0), v)$ is the set of points w such that $w \cdot v = 0$.

In general the set of points w such that $w \cdot v = c$ (some constant) is a line perpendicular to the vector v. some distance $c/|v|$ from the origin. So if v is normalized $w \cdot v$ is the signed distance from the perpendcular through the origin.

More on the dot product

Given two vectors v and w we have that $v \cdot w = |v||w| \cos(\alpha)$ where α is the angle between v and w. This means that we can compute the angle between two vectors v and w by computing

$$\alpha = cos^{-1}\left(\frac{v}{|v|}\frac{w}{|w|}\right)$$

Representing Lines: Two Standard Ways

$$Ax + By = C$$

where A, B and C are constants is the equation of a line. (A and B cannot both be zero.) This is the same as saying that $(A, B) \cdot (x, y) = C$.

Other method: Given a pair of distinct points p and q, it's the unique line through those two points.

Can convert between the two representations.
Take the line through (p, q). Let $v =$rot90cc$(p - q)$.
Now

$$v \cdot (x, y) = v \cdot p$$

is its dot product representation.

Distance to a line

Take the line L: $Ax + By = C$. Where the vector (A, B) is normalized, i.e. $A^2 + B^2 = 1$.)
Then for any point p, $(A, B) \cdot p$ is its signed distance from the line L.

Take the line L: through points a and b. Let $v = (b - a)/|b - a|$.
Then the signed distance from p to L is

$$(p - a) \times v$$

Line Side Test

Given a line through points p and q, from $p \to q$ which side of that directed line is a on? LEFT, ON, RIGHT?

Let $v = q - p$, and $w = a - p$

Take $v \times w$.

If > 0 *a is to the LEFT of* $p \to q$
if < 0 *a is to the RIGHT of* $p \to q$
if $= 0$ *q is ON* $p \to q$

Convex Combination of Points

Given $p_1, p_2, \ldots, p_n$ a convex combination of them is $\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n$. Where the αs are non-negative scalars that sum to 1.

Example $n = 2$: It's the segment connecting the two points.

For $n = 3$ it fills the triangle. (show general case)

Convex Sets

A set C of points in R^d is convex if for any points p, q in C the convex combination of p and q is also in C.
(Examples of convex and non-convex sets)

The intersection of any two convex sets is convex (proof trivial from the definition)

Note: The half plane is convex, so the intersection of half planes is convex (i.e. a convex polytope, i.e. the feasible region of an LP.)

Part 2: The Convex Hull Problem

Given a set of points $p_1, \ldots, p_n$ in 2D, compute the convex hull of the points. I.e. the bounary of the convex combination of the points.

Intuitively the convex hull can be computed with the following analog computer: Put nails in a board at the points $p_1, \ldots, p_n$. Now let a rubber band stretched around all of them snap in. The points it touches, and the order of them around that polygon is the convex hull of $p1, \ldots, p_n$.

Convex Hull Contd.

Input: A set of points $p_1, \ldots, p_n$. We assume for clarity that no three of them are colinear.

Output: A sequence from among $p_1, \ldots, p_n$ of the convex hull of the points in counter-clockwise order.

An $O(n^3)$ algorithm

Try all pairs to see if it's a convex hull boundary. Stitch the selected ones together.

An $O(n^2)$ algorithm

Graham Scan: an $O(n \log n)$ algorithm

An $O(n \log n)$ lower bound for convex hull