Warm-up: DFS Graph Search

In HW1 Q4.1, why was the answer $S \rightarrow C \rightarrow G$, not $S \rightarrow A \rightarrow C \rightarrow G$?

After all, we were doing DFS and breaking ties alphabetically.
AI: Representation and Problem Solving

Informed Search

Instructors: Pat Virtue & Stephanie Rosenthal

Slide credits: CMU AI, http://ai.berkeley.edu
Announcements

Assignments:

- P0: Python & Autograder Tutorial
  - Due Thu 1/24, 10 pm
- HW2 (written)
  - Due Mon 1/28, 10 pm
  - No slip days! Up to 24 hours late, 50% penalty
- P1: Search & Games
  - Released Thu 1/24, 10 pm
  - Due Thu 2/7, 10 pm

Pat is out next week (AAAI/EAAI conference)

- Stephanie lecturing next week
A Note on CS Education

Formative vs Summative Assessment

- https://www.cmu.edu/teaching/assessment/basics/formative-summative.html
Warm-up: DFS Graph Search

In HW1 Q4.1, why was the answer S->C->G, not S->A->C->G?

After all, we were doing DFS and breaking ties alphabetically.
function TREE_SEARCH(problem) returns a solution, or failure

initialize the frontier as a specific work list (stack, queue, priority queue)
add initial state of problem to frontier
loop do
    if the frontier is empty then
        return failure
    choose a node and remove it from the frontier
    if the node contains a goal state then
        return the corresponding solution
    for each resulting child from node
        add child to the frontier
function GRAPH_SEARCH(problem) returns a solution, or failure

initialize the explored set to be empty
initialize the frontier as a specific work list (stack, queue, priority queue)
add initial state of problem to frontier

loop do
  if the frontier is empty then
    return failure
  choose a node and remove it from the frontier
  if the node contains a goal state then
    return the corresponding solution
  add the node state to the explored set
  for each resulting child from node
    if the child state is not already in the frontier or explored set then
      add child to the frontier
function UNIFORM-COST-SEARCH(problem) returns a solution, or failure
initialize the explored set to be empty
initialize the frontier as a priority queue using node path_cost as the priority
add initial state of problem to frontier with path_cost = 0
loop do
  if the frontier is empty then
    return failure
  choose a node and remove it from the frontier
  if the node contains a goal state then
    return the corresponding solution
  add the node state to the explored set
  for each resulting child from node
    if the child state is not already in the frontier or explored set then
      add child to the frontier
    else if the child is already in the frontier with higher path_cost then
      replace that frontier node with child
Recall: Breadth-First Search (BFS) Properties

What nodes does BFS expand?
- Processes all nodes above shallowest solution
- Let depth of shallowest solution be $s$
- Search takes time $O(b^s)$

How much space does the frontier take?
- Has roughly the last tier, so $O(b^s)$

Is it complete?
- $s$ must be finite if a solution exists, so yes!

Is it optimal?
- Only if costs are all the same (more on costs later)
Size/cost of Search Trees

See Piazza post:
https://piazza.com/class/jpst41cibre86kp?cid=60
Recall: Breadth-First Search (BFS) Properties

What nodes does BFS expand?
- Processes all nodes above shallowest solution

Let depth of shallowest solution be $s$
- Search takes time $O(b^s)$

How much space does the frontier take?
- Has roughly the last tier, so $O(b^s)$
Uniform Cost Search (UCS) Properties

What nodes does UCS expand?
- Processes all nodes with cost less than cheapest solution
- If that solution costs $C^*$ and step costs are at least $\varepsilon$, then the “effective depth” is roughly $\frac{C^*}{\varepsilon}$
- Takes time $O(b^{C^*/\varepsilon})$ (exponential in effective depth)

How much space does the frontier take?
- Has roughly the last tier, so $O(b^{C^*/\varepsilon})$

Is it complete?
- Assuming best solution has a finite cost and minimum step cost is positive, yes!

Is it optimal?
- Yes! (Proof via A*)
Uniform Cost Issues

Strategy:
- Explore (expand) the lowest path cost on frontier

The good:
- UCS is complete and optimal!

The bad:
- Explores options in every “direction”
- No information about goal location

We’ll fix that today!

[Demo: contours UCS empty (L3D1)]
[Demo: contours UCS pacman small maze (L3D3)]
Demo Contours UCS Empty
Demo Contours UCS Pacman Small Maze
Uninformed vs Informed Search
Informed Search

- Heuristics
- Greedy Search
- A* Search
Informed Search
Search Heuristics

A heuristic is:

- A function that *estimates* how close a state is to a goal
- Designed for a particular search problem
- Examples: Manhattan distance, Euclidean distance for pathing
Example: Euclidean distance to Bucharest

- Arad: 366 (Mehadia: 241)
- Bucharest: 0 (Neamt: 234)
- Craiova: 160 (Oradea: 380)
- Drobeta: 242 (Pitesti: 100)
- Eforie: 161 (Rimnicu Vilcea: 193)
- Fagaras: 176 (Sibiu: 253)
- Giurgiu: 77 (Timisoara: 329)
- Hirsova: 151 (Urziceni: 80)
- Iasi: 226 (Vaslui: 199)
- Lugoj: 244 (Zerind: 374)

h(state) → value
Effect of heuristics

Guide search *towards the goal* instead of *all over the place*
Greedy Search
Greedy Search

Expand the node that seems closest... (order frontier by \( h \))

What can possibly go wrong?

Sibiu-Fagaras-Bucharest =
\[ 99 + 211 = 310 \]

Sibiu-Rimnicu Vilcea-Pitesti-Bucharest =
\[ 80 + 97 + 101 = 278 \]
Greedy Search

Strategy: expand a node that \textit{seems} closest to a goal state, according to $h$

Problem 1: it chooses a node even if it’s at the end of a very long and winding road

Problem 2: it takes $h$ literally even if it’s completely wrong
Demo Contours Greedy (Empty)
Demo Contours Greedy (Pacman Small Maze)
A* Search
**A* Search**

- **UCS**
  - $S \rightarrow n$
  - $g(n)$

- **Greedy**
  - $h(n) \sim n \rightarrow G$

- **A***
  - $f(n) = g(n) + h(n)$
Combining UCS and Greedy

Uniform-cost orders by path cost, or **backward cost** $g(n)$

Greedy orders by goal proximity, or **forward cost** $h(n)$

A* Search orders by the sum: $f(n) = g(n) + h(n)$

Example: Teg Grenager
Is A* Optimal?

What went wrong?

*Actual* bad goal cost < *estimated* good goal cost

We need estimates to be less than actual costs!
The Price is Wrong...

Closest bid without going over...

https://www.youtube.com/watch?v=9B0ZKRurC5Y
Admissible Heuristics
Admissible Heuristics

A heuristic $h$ is *admissible* (optimistic) if:

$$0 \leq h(n) \leq h^*(n)$$

where $h^*(n)$ is the true cost to a nearest goal

Example:

Coming up with admissible heuristics is most of what’s involved in using A* in practice.
Optimality of A* Tree Search
Optimality of A* Tree Search

Assume:

- $A$ is an optimal goal node
- $B$ is a suboptimal goal node
- $h$ is admissible

Claim:

$A$ will be chosen for exploration (popped off the frontier) before $B$
Optimality of A* Tree Search: Blocking

Proof:
Imagine \( B \) is on the frontier
Some ancestor \( n \) of \( A \) is on the frontier, too
(Maybe the start state; maybe \( A \) itself!)
Claim: \( n \) will be explored before \( B \)

1. \( f(n) \) is less than or equal to \( f(A) \)

\[
f(n) = g(n) + h(n) \quad \text{Definition of } f\text{-cost}
\]
\[
f(n) \leq g(A) \quad \text{Admissibility of } h
\]
\[
g(A) = f(A) \quad h = 0 \text{ at a goal}
Optimality of A* Tree Search: Blocking

Proof:
Imagine \( B \) is on the frontier

Some ancestor \( n \) of \( A \) is on the frontier, too
(Maybe the start state; maybe \( A \) itself!)

Claim: \( n \) will be explored before \( B \)

1. \( f(n) \) is less than or equal to \( f(A) \)
2. \( f(A) \) is less than \( f(B) \)

\[ g(A) < g(B) \]
\[ f(A) < f(B) \]

Suboptimality of \( B \)
\( h = 0 \) at a goal
Optimality of A* Tree Search: Blocking

Proof:
Imagine \( B \) is on the frontier
Some ancestor \( n \) of \( A \) is on the frontier, too
(Maybe the start state; maybe \( A \) itself!)

Claim: \( n \) will be explored before \( B \)
1. \( f(n) \) is less than or equal to \( f(A) \)
2. \( f(A) \) is less than \( f(B) \)
3. \( n \) is explored before \( B \)

All ancestors of \( A \) are explored before \( B \)
\( A \) is explored before \( B \)
A* search is optimal
Properties of A*

Uniform-Cost

A*
UCS vs A* Contours

Uniform-cost expands equally in all “directions”

A* expands mainly toward the goal, but does hedge its bets to ensure optimality
Demo Contours (Empty) -- UCS
Demo Contours (Empty) – A*
Demo Contours (Pacman Small Maze) – A*
Comparison

Greedy  Uniform Cost  A*
Demo Empty Water Shallow/Deep – Guess Algorithm
A* Search Algorithms

A* Tree Search

- Same tree search algorithm as before but with a frontier that is a priority queue using priority $f(n) = g(n) + h(n)$

A* Graph Search

- Same as UCS graph search algorithm but with a frontier that is a priority queue using priority $f(n) = g(n) + h(n)$
function UNIFORM-COST-SEARCH(problem) returns a solution, or failure
initialize the explored set to be empty
initialize the frontier as a priority queue using g(n) as the priority
add initial state of problem to frontier with priority \( g(S) = 0 \)
loop do
    if the frontier is empty then
        return failure
    choose a node and remove it from the frontier
    if the node contains a goal state then
        return the corresponding solution
    add the node state to the explored set
    for each resulting child from node
        if the child state is not already in the frontier or explored set then
            add child to the frontier
        else if the child is already in the frontier with higher g(n) then
            replace that frontier node with child
function A-STAR-SEARCH(problem) returns a solution, or failure
initialize the explored set to be empty
initialize the frontier as a priority queue using \( f(n) = g(n) + h(n) \) as the priority
add initial state of problem to frontier with priority \( f(S) = 0 + h(S) \)
loop do
  if the frontier is empty then
    return failure
  choose a node and remove it from the frontier
  if the node contains a goal state then
    return the corresponding solution
  add the node state to the explored set
  for each resulting child from node
    if the child state is not already in the frontier or explored set then
      add child to the frontier
    else if the child is already in the frontier with higher \( f(n) \) then
      replace that frontier node with child
A* Applications

Pathing / routing problems
Resource planning problems
Robot motion planning
Language analysis
Video games
Machine translation
Speech recognition
...

Image: maps.google.com
Creating Heuristics

YOU GOT
HEURISTIC UPGRADE!
Creating Admissible Heuristics

Most of the work in solving hard search problems optimally is in coming up with admissible heuristics.

Often, admissible heuristics are solutions to \textit{relaxed problems}, where new actions are available.
Example: 8 Puzzle

What are the states?
How many states?
What are the actions?
How many actions from the start state?
What should the step costs be?
8 Puzzle I

Heuristic: Number of tiles misplaced

Why is it admissible?

\[ h(\text{start}) = 8 \]

This is a relaxed-problem heuristic

---

Start State:

\[
\begin{array}{ccc}
7 & 2 & 4 \\
5 & 1 & \text{ } \\
8 & 3 & 1
\end{array}
\]

Goal State:

\[
\begin{array}{ccc}
1 & 2 & \text{ } \\
3 & 4 & 5 \\
6 & 7 & 8
\end{array}
\]

---

Average nodes expanded when the optimal path has...

<table>
<thead>
<tr>
<th>Steps</th>
<th>UCS</th>
<th>A*TILES</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>112</td>
<td>13</td>
</tr>
<tr>
<td>8</td>
<td>6,300</td>
<td>39</td>
</tr>
<tr>
<td>12</td>
<td>3.6 x 10^6</td>
<td>227</td>
</tr>
</tbody>
</table>

Statistics from Andrew Moore
8 Puzzle II

What if we had an easier 8-puzzle where any tile could slide any direction at any time, ignoring other tiles?

Total Manhattan distance

Why is it admissible?

h(start) = 3 + 1 + 2 + ... = 18
Combining heuristics

Dominance: \( h_a \geq h_c \) if
\[ \forall n \ h_a(n) \geq h_c(n) \]

- Roughly speaking, larger is better as long as both are admissible
- The zero heuristic is pretty bad (what does A* do with \( h=0 \)?)
- The exact heuristic is pretty good, but usually too expensive!

What if we have two heuristics, neither dominates the other?
- Form a new heuristic by taking the max of both:
  \[ h(n) = \max( h_a(n), h_b(n) ) \]
- Max of admissible heuristics is admissible and dominates both!
Optimality of A* Graph Search
A* Tree Search

State space graph

Search tree

S (0+2)

A (1+4) C (3+1)

C (2+1) G (6+0)

G (5+0)

h=0

h=1

h=2

h=3

h=4
What paths does A* graph search consider during its search?

A) \( S, S-A, S-C, S-C-G \)
B) \( S, S-A, S-C, S-A-C, S-C-G \)
C) \( S, S-A, S-A-C, S-A-C-G \)
A* Graph Search

What does the resulting graph tree look like?

A)

B)

C & D)
A* Graph Search Gone Wrong?

State space graph

Search tree

Simple check against explored set blocks C
Fancy check allows new C if cheaper than old but requires recalculating C’s descendants
Admissibility of Heuristics

Main idea: Estimated heuristic values ≤ actual costs

- Admissibility:
  
  heuristic value ≤ actual cost to goal

  \[ h(A) \leq \text{actual cost from A to G} \]
Consistency of Heuristics

Main idea: Estimated heuristic costs ≤ actual costs

- **Admissibility:**
  
  heuristic cost ≤ actual cost to goal

  \[ h(A) \leq \text{actual cost from } A \text{ to } G \]

- **Consistency:**

  “heuristic step cost” ≤ actual cost for each step

  \[ h(A) - h(C) \leq \text{cost}(A \text{ to } C) \]

  triangle inequality

  \[ h(A) \leq \text{cost}(A \text{ to } C) + h(C) \]

Consequences of consistency:

- The f value along a path never decreases
- A* graph search is optimal
Optimality of A* Graph Search

Sketch: consider what A* does with a consistent heuristic:

- Fact 1: In tree search, A* expands nodes in increasing total f value (f-contours)

- Fact 2: For every state s, nodes that reach s optimally are explored before nodes that reach s suboptimally

- Result: A* graph search is optimal
Optimality

Tree search:
▪ A* is optimal if heuristic is admissible
▪ UCS is a special case (h = 0)

Graph search:
▪ A* optimal if heuristic is consistent
▪ UCS optimal (h = 0 is consistent)

Consistency implies admissibility

In general, most natural admissible heuristics tend to be consistent, especially if from relaxed problems
A*: Summary
A*: Summary

A* uses both backward costs and (estimates of) forward costs

A* is optimal with admissible / consistent heuristics

Heuristic design is key: often use relaxed problems