ZACHARY KARATE CLUB

34 vertices (karatekas), 78 edges (friendships)

networkkarate.tumblr.com
Vertices = people, edges = Friendships

#vertices $n = 10^9$, #edges $m = 10^{12}$
Kidney Exchange

Vertices = patient-donor pairs, edges = compatibility

UNOS pool, Dec 2010 [Courtesy John Dickerson, CMU]

World Wide Web

Vertices = pages, edges = hyperlinks

If your problem has a graph, great. If not, try to make it have a graph!
**TYPES OF GRAPHS**

- Simple
- Undirected Graphs
- Directed Graphs
- General Graphs
- "parallel edges"
- "self-loops"

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**RETRONYM**

- Acoustic Guitar
- Electric Guitar

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**BASIC DEFINITIONS**

- A graph $G$ is a pair:
  - $V$ is the set of vertices/nodes; $|V| = n$
  - $E$ is the set of edges; $|E| = m$
- Each edge is a pair $\{u, v\}$, where $u \neq v$
- Example:
  - $V = \{a, b, c, d\}$
  - $E = \{\{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}\}$
**EDGE CASES**

- A graph with no edges is called an empty graph
- Example:
  - $V = \{1, 2, 3, 4\}$
  - $E = \emptyset$

**THE NULL GRAPH**

**IS THE NULL-GRAF A POINTLESS CONCEPT?**

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**ABSTRACT**

The graph with no points and no lines is discussed critically. Arguments for and against its official admission as a graph are presented. This is accompanied by an extensive survey of the literature. Paradoxical properties of the null-graph are noted. No conclusion is reached.

**THE NULL GRAPH**

_Figure 1. The Null Graph_
**Mr. Vertex’s Neighborhood**

- If \( \{u, v\} \in E \), \( u \) is a neighbor of \( v \)
- The neighborhood \( N(u) \) of \( u \) is \( \{v \in V \mid \{u, v\} \in E\} \)
- The degree \( \deg(u) \) of \( u \) is \( |N(u)| \)

\[ N(b) = \{a, c\} \]
\[ \deg(b) = 2 \]

**Theorem:** \( \sum_{u \in V} \deg(u) = 2m \)

**Proof:**
- Each vertex places a token on each of its edges
- The number of tokens is \( \sum_{u \in V} \deg(u) \)
- Each edge has exactly two tokens placed on it
- The number of tokens is \( 2m \) \( \blacksquare \)

2 + 2 + 3 + 1 = 2 \cdot 4

**Facebook, revisited**

\#vertices \( n = 10^9 \), \#edges \( m = 10^{12} \)
**REGULAR GRAPHS**

- A graph is $d$-regular if all nodes have degree $d$
- The empty graph is 0-regular
- 1-regular graph is called a perfect matching
- Poll 1: How many 2-regular graphs with $V = \{a, b, c, d\}$ are there?

| 1 | 3 | 6 | 12 |

**3-REGULAR GRAPHS**

There are lots and lots of possibilities

**CONNECTEDNESS**

- Graph $G$ is connected if for all $u, v \in V$ there is a path between $u$ and $v$

This 11-vertex graph is not connected
It has 3 connected components
CONNECTEDNESS

What is the minimum number of edges needed to make a connected 27-vertex graph?

\[ n = 1 \quad \text{Done} \quad m = 0 \]
\[ n = 2 \quad m = 1 \quad \text{necessary and sufficient} \]
\[ n = 3 \quad m = 2 \quad \text{necessary and sufficient} \]
\[ n = 4 \quad m = 3 \quad \text{necessary and sufficient} \]

\[ n - 1 \text{ edges are always sufficient to connect an } n\text{-vertex graph} \]

“star graph”

“path graph”

“something else”
Theorem: $n - 1$ edges are also necessary to connect an $n$-vertex graph

Proof:
- If $G$ has $k$ connected components, and $G'$ is formed from $G$ by adding an edge, then $G'$ has at least $k - 1$ components
- Add edges one by one to obtain a single connected component, need at least $n - 1$ steps.

ACYCLIC GRAPHS

Poll 2: Assume that $G$ is connected. Then:
1. $m = n - 1 \Rightarrow G$ is acyclic
2. $G$ is acyclic $\Rightarrow m = n - 1$
3. $G$ is acyclic $\Rightarrow m = n - 1$
4. Incomparable

TREES

A tree is a connected acyclic graph
**Graph Theory Haiku**

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**Ore’s Theorem**

- A Hamiltonian cycle in \( G \) is a cycle that visits every \( v \in V \) exactly once (see Lect. 9).
- Theorem [Ore, 1960]: Let \( G \) be a graph on \( n \geq 3 \) vertices such that \( \deg(u) + \deg(v) \geq n \) for any \( u, v \in V \) that are not neighbors, then \( G \) contains a Hamiltonian Cycle.

**Proof of Ore’s Theorem**

- Color the edges of \( G \) blue, add red edges to form a complete graph, and choose a Hamiltonian Cycle \( \mathcal{C} \).
- If \( \mathcal{C} \) is not completely blue, will find \( \mathcal{C'} \) with more blue edges.
PROOF OF ORE’S THEOREM

• Let \( (a, b) \) be a red edge in \( C \)
• Let \( S \) be the successors of \( N(a) \) on \( C \)
• \( \deg(b) \geq n - \deg(a) \)
  \[ = |V| - |N(a)| \]
  \[ = |V| - |S| \]
  \[ > |V \setminus (S \cup \{b\})| \]
• So \( b \) is a neighbor of \( c \in S \)
• We can find a bluer cycle \( \Box \)

SUMMARY

• Terminology:
  - Regular graph
  - Connected graph
  - Neighborhood, degree
  - Hamiltonian cycle
• Theorems:
  - If \( G \) is connected
    \( |E| = n - 1 \implies \text{acyclic} \)
  - \( \sum_{u \in V} \deg(u) = 2m \)