These Decidable Definitions Have Undecidable Ends

- A **decider** is a TM that halts on all inputs.
- A language $L$ is **undecidable** if there is no TM $M$ that halts on all inputs such that $M(x)$ accepts if and only if $x \in L$.
- A language $A$ **reduces** to $B$ if it is possible to decide $A$ using an algorithm that decides $B$ as a subroutine. Denote this as $A \leq B$ (read: $B$ is at least as hard as $A$).
- Countability cheat sheet: You are given a set $A$. Is it countable or uncountable?

\[
\begin{array}{c|c}
|A| \leq |\mathbb{N}| & (A \text{ is countable}) \\
\text{– Show directly an injection from } A \text{ to } \mathbb{N} & (A \hookrightarrow \mathbb{N}) \text{ or a surjection from } \mathbb{N} \text{ onto } A \\
& (\mathbb{N} \twoheadrightarrow A) \\
\text{– Show } |A| \leq |B|, \text{ where } B \text{ is one of } \mathbb{Z}, \mathbb{Z} \times \mathbb{Z}, \mathbb{Q}, \Sigma^*, Q[x], \text{ etc.} & |A| > |\mathbb{N}| (A \text{ is uncountable}) \\
\text{– Show directly using a diagonalization argument.} & \\
\text{– Show that } |\{0,1\}^\infty| \leq |A|, \text{ i.e. an injection from } \{0,1\}^\infty \text{ to } A.
\end{array}
\]

*This one is important and very powerful*

Counting sheep

For each set below, determine if it is countable or not. Prove your answers.

(a) $S = \{a_1 a_2 a_3 \ldots \in \{0,1\}^\infty \mid \forall n \geq 1 \text{ the string } a_1 \ldots a_n \text{ contains more } 1\text{'s than } 0\text{'s.}\}$

(b) $\Sigma^*$, where $\Sigma$ is an alphabet that is allowed to be countably infinite (e.g., $\Sigma = \mathbb{N}$).

Doesn’t Look Like Anything (Decidable) To Me

Prove that the following languages are undecidable (below, $M$, $M_1$, $M_2$ refer to TMs).

(a) $\text{REGULAR} = \{\langle M \rangle : L(M) \text{ is regular}\}$.

(b) $\text{TOTAL} = \{\langle M \rangle : M \text{ halts on all inputs}\}$.

(c) $\text{DOLORES} = \{\langle M_1, M_2 \rangle : \exists w \in \Sigma^* \text{ such that both } M_1(w) \text{ and } M_2(w) \text{ accept}\}$.
(Extra) Lose All Scripted Responses. Improvisation Only

Let $\text{FINITE} = \{\langle M \rangle : M \text{ is a TM and } L(M) \text{ is finite}\}$.

Show that $\text{TOTAL} \leq \text{FINITE}$.

(Bonus) The Maize is not Meant For You

Josh Corn is trying to write a program $P$ such that given a natural number $n$, $P(n)$ is the most number of steps a TM on $n$ states can take before halting. Show that this is not possible.