Consider the problem of proving that $\forall n \geq 0, 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$ by induction.

Define the statement $S_n = "1 + 2 + \ldots + n = \frac{n(n+1)}{2}"$. We want to prove $\forall n \geq 0, S_n$.

1  An Inductive Proof

**Base Case:** $\frac{0(0+1)}{2} = 0$, and hence $S_0$ is true.

**I.H.:** Assume that $S_k$ is true for some $k \geq 0$.

**Inductive Step:** We want to prove the statement $S(k+1)$. Note that

\[
1 + 2 + \ldots + k + (k + 1) = \frac{k(k+1)}{2} + (k + 1) \quad \text{(by I.H.)}
\]

\[
= (k + 1)\left(\frac{k}{2} + 1\right)
\]

\[
= (k+1)(k+2)\cdot
\]

And hence $S_{k+1}$ is true.

2  Common Errors and Pitfalls

1. *(S_n is a statement, not a value)* You cannot make statements like $S_k + (k + 1) = S_{k+1}$, much the same as you cannot add $k$ to the statement “The earth is round”.

   **Mistake:**
   
   **I.H.:** Assume that $S_k$ is true.
   
   **Inductive Step:**
   
   \[
   \sum_{i=1}^{k+1} i = k + 1 + \sum_{i=1}^{k} i
   \]
   \[
   = k + 1 + S_k
   \]
   \[
   = \ldots
   \]

   Logical propositions like $S_k$ can’t be added to numbers. Please don’t equate propositions and arithmetic formulas.
2. *(Proof going the Wrong Way)* Make sure you use $S_k$ to prove $S_{k+1}$, and not the other way around. Here is a common (wrong!) inductive step:

**Mistake:**
Inductive Step:

$$
1 + 2 + \ldots + k + (k + 1) = \frac{(k + 1)(k + 2)}{2} \\
k(k + 1)/2 + (k + 1) = \frac{(k + 1)(k + 2)}{2} \\
\frac{(k + 1)(k + 2)}{2} = \frac{(k + 1)(k + 2)}{2}.
$$

The proof above starts off with $S_{k+1}$ and ends using $S_k$ to prove an identity, which does not prove anything. Please make sure you do not assume $S_{k+1}$ in an effort to prove it!

3. *(Assuming too much)* Make sure you don’t assume *everything* in the I.H.

**Mistake:**

I.H.: Assume that $S_k$ is true for all $k$.

You want to prove the statement $S_n$ true for all $n$, and if you assume it is true, there is nothing left to prove! (Remember that the “$S_n$ is true for all $n$” is the same as saying “$S_k$ is true for all $k$”.)

**Correct Way:**

I.H.: Assume that $S(k)$ is true for some $k$.

or, if you want to use all-previous (“strong”) induction

I.H.: Assume for some $k$ that $S(j)$ is true for all $j \leq k$.

4. *(The case of the missing $n$)* Consider the following I.H. and inductive step:

**Mistake:**

I.H.: Assume that $S_k$ is true for all $k \leq n$.

Inductive Step: We want to prove $S_{k+1}$.

What is $k$? Where has $n$ disappeared? The induction hypothesis is saying in shorthand that $S_1, S_2, \ldots, S_{n-1}, S_n$ are all true for some $n$. Note that rewriting the I.H. in this way shows that $k$ was a red herring: you really want to prove $S_{n+1}$, not $S_{k+1}$.

**Correct Way:**

I.H.: Assume that $S_k$ is true for all $k \leq n$.

Inductive Step: We want to prove $S_{n+1}$. 
5. *(Extra stuff in the I.H.)* Consider the following I.H.

**Mistake:**

I.H.: Assume that $S_k$ is true for all $k \leq n$. **Then $S_{n+1}$**.

Note that entire thing has been made part of the hypothesis, including the bolded part. The second part “Then $S_{n+1}$” is what you want to show in the inductive step; it is *not* part of the induction hypothesis. You need to distinguish between the *Claim* and the *Induction Hypothesis*. The Claim is the statement you want to prove (i.e., $\forall n \geq 0, S_n$), whereas the Induction Hypothesis is an *assumption* you make (i.e., $\forall 0 \leq k \leq n, S_n$), which you use to prove the next statement (i.e., $S_{n+1}$). The I.H. is an assumption which might or might not be true (but if you do the induction right, the induction hypothesis will be true).

**Correct Way:**

I.H.: Assume that $S_k$ is true for all $k \leq n$.

6. *(The Wrong Base Case.)* Note that you want to prove $S_0$, $S_1$, etc., and hence the base case should be $S_0$.

**Mistake:**

Base Case: $\frac{(1+1)}{2} = 1$, and hence $S_1$ is true.

Even if the rest of the proof works fine, you would have shown that $S_1, S_2, S_3, \ldots$ are all correct. You haven’t shown that $S_0$ is true.

7. *(Assuming too little: Too few Base Cases.)*

Suppose you were given a function $X(n)$ and need to show that the statement $S_n$ that “the Fibonacci number $F_n = X(n)$” for all $n \geq 0$.

**Mistake:**

Base Case: for $n = 0$, $F_0 = X(0)$ blah blah. Hence $S_0$ is true.
I.H.: Assume that $S_k$ is true for all $k \leq n$.

**Induction Step:** Now $F_n = F_{n-1} + F_{n-2} = X(n-1) + X(n-2)$ (because $S_{n-1}$ and $S_{n-2}$ are both true), etc.

If you are using $S_{n-1}$ and $S_{n-2}$ to prove $T(n)$, then you better prove the base case for $S_0$ and $S_1$ in order to prove $S_2$. Else you have shown $S_0$ is true, but have no way to prove $S_1$ using the above proof—$S_0$ is not a base case, and to use induction, we’d need $S_0$ and $S_{-1}$. But there is no $S_{-1}$!!!

Remember the domino principle: the above induction uses the fact that “if two consecutive dominoes fall, the next one will fall”. To now infer that *all* the dominoes fall, you must show that the first two dominoes fall. And hence you need two base cases.