1. (SOLO)

(a) Given a Boolean circuit $C$ with $n$ input gates, let $f_C : \{0,1\}^n \to \{0,1\}$ denote the function that $C$ computes. Show that the language

$$\text{CIRCUIT-NEQ} = \{\langle C_1, C_2 \rangle : C_1 \text{ and } C_2 \text{ are Boolean circuits and } f_{C_1} \neq f_{C_2} \}$$

is in NP.

(b) Let $L$ be the set of all words in $\{0,1\}^*$ corresponding to the binary representation of a composite number. Prove that $L \in \text{NP}$.

2. (SOLO) Show that $\text{NP}$ is closed under the union, intersection, concatenation and star operations. That is, show that if languages $L_1$ and $L_2$ are in $\text{NP}$, then so are $L_1 \cup L_2$, $L_1 \cap L_2$, $L_1 L_2$ and $L_1^*$.

3. (SOLO) Let $K \subseteq \{0,1\}^*$ consist of all strings containing at least two 1’s. Describe a circuit family of size $O(n)$ computing $K$ (a proof that your circuit family correctly computes $K$ is not required, but it should be clear from your construction that it is indeed correct). Prove that the size bound is as required. (Hint: One approach is to think recursive.)

4. (GROUP) In the regular Stable Matching problem, we assumed that participants prefer to be matched over being single. In this question, we will remove this condition, as it may not be always a realistic assumption. We modify the original Stable Matching problem so that everyone now has a list of preferences which can potentially be incomplete (i.e., the list may not include every possible person from the opposite set of participants). As before, a person prefers being matched with someone on their list over being single. But a person prefers being single over being matched with someone not on their list. In this modification, the notion of a stable matching is a matching (not-necessarily perfect) in which:

- There is no pair $(m, w)$ where $m$ and $w$ are not matched to each other, but they prefer each other over their current situation (i.e., their current match or being single).
- There is no matched person who prefers being single over being matched with their current partner.

(a) Does a stable matching always exist in this setting? Prove your answer.

(b) Is it true that if a person is single in one stable matching, then he or she is single in all stable matchings (i.e., is it true that there is no hope for some people)? Prove your answer.

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1 A number is composite if it is strictly greater than 1 and is not prime.