15-251: Great Theoretical Ideas In Computer Science

Recitation 9 Solutions

Quiz

Note that there is a quiz at the beginning of lecture tomorrow, which will cover graphs and automata.

Graph Theory

1. *Show that every graph G has a (simple) path of length δ, where δ is the minimum degree over all the vertices of G. A path is called simple if it does not contain any repeated vertices, and the length of a path is the number of edges in the path. Show that every graph has a cycle of length at least $\delta + 1$.*

   Solution: Let P be the longest simple path in G, with vertices $(v_1, v_2, \cdots, v_k)$, where all the v_i's are distinct. Note that v_1 and v_k cannot be adjacent to any vertices not on the path P, as otherwise the path could be extended to get a longer path. Since every vertex in the graph has degree at least δ, all of the δ vertices adjacent to v_1 must all lie on the path P. This means that P has at least $\delta + 1$ nodes, and at least δ edges.

   Moreover, if v_i is the last vertex on the path with an edge to v_1, then the index i must be at least δ. The cycle $v_1 v_2 \cdots v_i v_1$ is a cycle of length $\delta + 1$.

2. *Prove that in a planar bipartite graph G with n vertices, there are at most $2n - 4$ edges. (This graph G is simple: it has no parallel edges or self-loops.)*

   Solution: Let f be a face of G. f must be bordered by at least 4 edges (otherwise it forms a 3-cycle, which cannot exist in a bipartite graph). On the other hand, each edge borders at most two faces. Counting the number of edge-face incidences, then, gives $4f \leq 2e$. Plugging this into Euler's formula, we get

   $$n - e + f = 2 \implies n - e + (e/2) \geq 2 \implies 2n - e \geq 4$$

   which is the same as $e \leq 2n - 4$.

DFAs

3. *Which of the following statements about regular languages are true? For every language claimed to be regular, contruct a DFA. For every language claimed to be non-regular, give a proof.*

   (a) *The empty set is a regular language.*
   Solution: True: the DFA with one non-accepting state recognizes it.

   (b) *The set of all strings over an alphabet Σ is a regular language.*
   Solution: True: the DFA with one *accepting* state recognizes it.

   (c) *If L is a regular language, then $L^c = \{w | w \notin L\}$ is a regular language.*
   Solution: True: given a DFA $M = (Q, \Sigma, \delta, q_0, F)$ that accepts L, swap the accepting and non-accepting states to get the DFA $M^c = (Q, \Sigma, \delta, q_0, Q \setminus F)$: note that any string that ended in a state in F is now rejecting instead of accepting, and any string that ended in a state in $Q \setminus F$ is now accepting instead of rejecting. Hence M^c accepts L^c.

   (d) *If L_1 is a regular language, and $L_2 \subset L_1$, then L_2 is regular.*
   Solution: False. Let $L_1 = \Sigma^*$, the set of *all* strings over the alphabet $\{a, b\}$, and L_2 be the language $\{a^n b^n\}$. As we showed above, L_1 is regular, but as we showed in lecture, L_2 is not.

(e) *If L_1 is a regular language, and $L_1 \subset L_2$, then L_2 is regular.*
Solution: False: Let L_1 be the empty set (which we showed above to be regular), and $L_2 = \{a^n b^n\}$ (which we know is not regular).

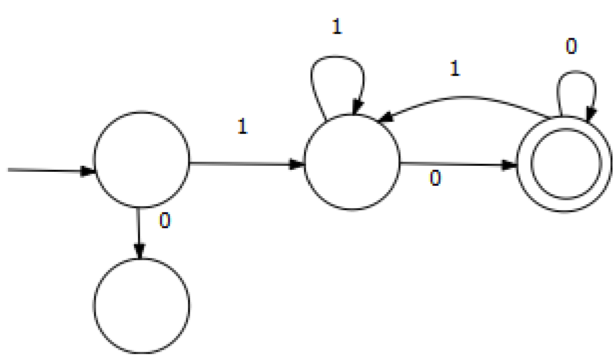
(f) *If $\Sigma = \{a, b, c\}$, then the language $L = \{w|$the number of occurrences of "ab" $=$ the number of occurrences of "ba"$\}$ is regular.*
Solution: False. Suppose there were a DFA M with n states that recognized L. Then, consider the behavior of M on the $n+1$ strings $abc, abcabc, abcabcabc, (abc)^i, (abc)^{n+1}$. By the pigeonhole principle, there must be two distinct values i and j such that the strings $(abc)^i$ and $(abc)^j$ cause M to reach the same state. Hence M cannot distinguish between the strings $S_1 = (abc)^i(bac)^i$ and $S_2 = (abc)^i(bac)^j$. However, note that the first string S_1 has i occurrences of ab and i occurrences of ba, while the second has i occurrences of ab and $j \neq i$ occurrences of ba. This contradicts the assumption that the DFA M (correctly) recognizes L. Therefore, there is no DFA that accepts L, so L is not regular.

Note: This is *not* the same problem as discussed in lecture: there, we had an alphabet $\{a, b\}$ whereas here we have the alphabet $\{a, b, c\}$.

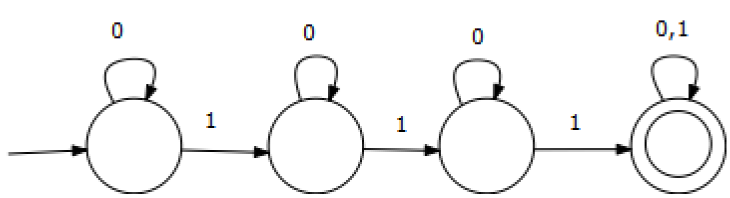
(g) *Fix the alphabet $\Sigma = \{(,)\}$ (the symbols are the two parentheses). Show that the language $L' = \{w \in \Sigma^* \mid$ the parentheses in w match$\}$ is regular.*
False. Suppose $L' = \{w \in \Sigma^* \mid$ the parentheses in w match$\}$ is regular. Then, there exists some DFA that accepts it. Let n be the number of states in the DFA.
Consider the $n+1$ strings of the form $($, $(($, $((($, $(^{n+1}$. Since there are more left strings than states, there must be two distinct values i and j such that $(^i$ and $(^j$ reach the same state. But now the two strings $(^i)^i$ and $(^j)^i$ both lead to the same state, yet the DFA must accept the former but reject the latter. Therefore, no DFA for L' exists, and L' is not regular.

4. For all of the below, the alphabet is $\Sigma = \{0, 1\}$. For each one, construct a DFA that accepts the language.

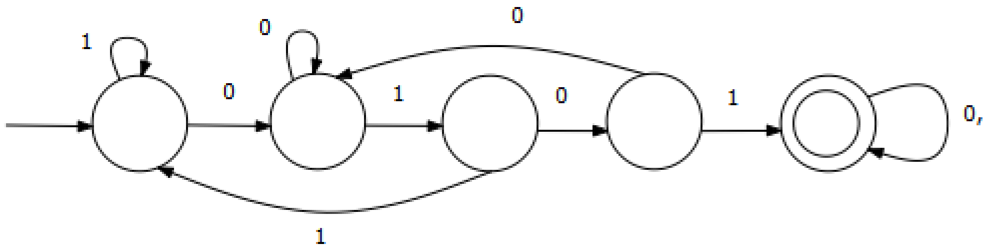
(a) $\{w|w$ begins with a 1 and ends with a 0$\}$
Solution:



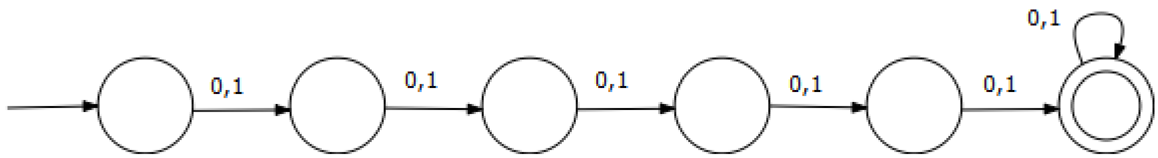
(b) $\{w|w$ contains at least three 1s$\}$
Solution:



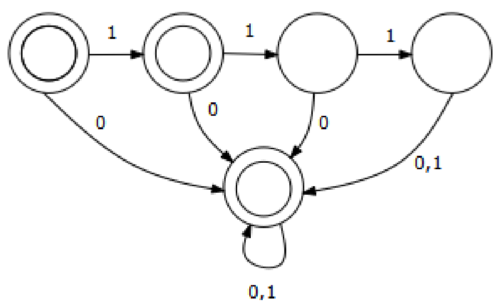
(c) $\{w | w$ contains the substring "0101"$\}$
Solution:



(d) $\{w |$the length of w is at least 5$\}$
Solution:



(e) $\{w | w$ is any string except "11" or "111"$\}$
Solution:



(f) $\{w | w$ contains at least two 0's and at most one 1$\}$
Solution: