

## 15-251: Great Theoretical Ideas In Computer Science

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### Recitation 6

#### Number Theory Statements

Below are some “if and only if” statements. Decide which directions (if any) of each statement are true. For implications that are true, give a short proof. For each that is false, give a counterexample. All letters below  $(n, x, y, \dots)$  represent integer values.

1.  $n$  is odd if and only if  $n^2 + 2n$  is odd.
2.  $\gcd(x, y) = 1$  if and only if  $\gcd(x^2, y^2) = 1$ .
3.  $x|yz$  if and only if  $x|y$  or  $x|z$ .
4.  $x|(y + z)$  if and only if  $x|y$  and  $x|z$ .
5.  $x|y$  if and only if  $x^2|y^2$ .
6.  $x|y$  if and only if  $x^3|y^2$ .

#### Modular Congruences

7. Find the smallest positive  $n$  such that  $n \equiv 4482341 \pmod{11}$ .
8. What is the units digit of  $7^{7^7}$ ?

#### Euclid's Algorithm

9. Find all solutions to  $17x + 47y = 4$ .

#### Number Theory Proofs

10. Let  $d$  be any positive integer not equal to 2, 5, or 13. Show that you can find distinct  $(a, b)$  in the set  $\{2, 5, 13, d\}$  such that  $ab - 1$  is not a perfect square.
11. Prove that every year (including leap years) has at least one Friday the 13th.

#### More Fun With Phi

Recall from lecture that  $\phi(n)$  is the number of positive integers less than or equal to  $n$  which are relatively prime to  $n$ .

12. Show that if  $\text{GCD}(a, b) = 1$ , then  $\phi(ab) = \phi(a)\phi(b)$ .
13. Show that if  $p$  is a prime number, then for all  $k \geq 1$ ,  $\phi(p^k) = p^{k-1}(p - 1)$ .