15-251: Great Theoretical Ideas In Computer Science

Recitation 5

Extra Credit Revisited

1. Devise a pair of dice which are six-sided cubes with *positive non-zero* integers on their faces, such that the distribution of the outcomes is exactly the same as for ordinary dice (the sum of the two dice being 2 is achieved in one way, the sum 3 is achieved in two ways, ..., 7 in six ways, ..., the sum 12 is achieved in one way), but which are different from ordinary dice.

Great Expectations

2. Which of the following statements are true in general:

   (a) For any two random variables $E[X+Y] = E[X] + E[Y]$.

   (b) For any two random variables $E[X \times Y] = E[X] \times E[Y]$.

   (c) For any random variable and a constant $c \in \mathbb{R}$, $E[c \cdot X] = c \cdot E[X]$.

   (d) For any two *independent* random variables $E[X/Y] = E[X]/E[Y]$.

3. The Can-Do candy company sells chocolate truffles that have either cherry or raspberry centers. Alice really wants to eat a cherry truffle, so she decides to eat truffles until she gets a cherry truffle—she stops once she has eaten such a truffle. You may assume that each time she picks a truffle, it is equally likely to be cherry or raspberry. What is the expected number of cherry truffles Alice eats? What is the expected number of raspberry truffles she eats?

4. Bob is greedier than Alice, and will eat truffles until he has eaten 5 cherry truffles. What is the expected number of raspberry truffles he eats before he stops?

5. The TAs decide to distribute cookies to the students in class. There are $52 = 2 \times 26$ students in lecture; the TAs have $156 = 6 \times 26$ cookies. Each cookie is in the shape of a letter of the alphabet, and there are 6 A's, 6 B's, ... 6 Z's. Each student gets 3 cookies which he/she eats. If the cookies are distributed completely randomly to the students, what is the expected number of cookies that are eaten by students whose first name begins with the letter corresponding to the cookie?

6. In the problem above, what is the expected number of students who get *at least* one cookie which matches their first name?

7. You toss independently a fair coin and you count the number of tosses until the first head appears. If this number is n, you receive 2^n dollars. What is the expected amount that you will receive?

Conditional Probability and Bayes' Rule

8. Suppose that with probability $\frac{2}{3}$, a randomly selected vehicle will pass inspection at a headlight inspection station. Assuming that successive vehicles pass or fail independently of one another, calculate the following probabilities.

   (a) P(all of the next three vehicles inspected pass)

   (b) P(exactly one of the next three inspected passes)

   (c) Given that at least one of the next three vehicles passes inspection, what is the probability that all three pass?

9. Suppose that an error in the dice manufacturing plant made it so that one out of every one hundered dice in production will always roll a six. If you pick a random die from the plant and roll it three times and each result is a six, what is the probability that the next roll will also be a six?