15-251: Great Theoretical Ideas In Computer Science

Recitation 5 Solutions

Extra Credit Revisited

1. Devise a pair of dice which are six-sided cubes with *positive non-zero* integers on their faces, such that the distribution of the outcomes is exactly the same as for ordinary dice (the sum of the two dice being 2 is achieved in one way, the sum 3 is achieved in two ways, ..., 7 in six ways, ..., the sum 12 is achieved in one way), but which are different from ordinary dice.

   First, we can represent the six-sided cube as the following generating function:

   $$D(x) = x^1 + x^2 + x^3 + x^4 + x^5 + x^6$$

   this captures the fact that there is exactly one way to get each of the options $1, 2, \ldots, 6$ using this die.
   Since we have two dice, the generating function for the sum of the two dice is:

   $$P(x) = (x^1 + x^2 + x^3 + x^4 + x^5 + x^6)^2$$

   Note that, if you expand this out, you will get that there is one way to get 2 (the coefficient of x^2 is 1), six ways to get 7 (the coefficient of x^7 is 6), etc.
   Therefore, the question asks for a different way to factor $P(x)$ into two polynomials $A(x) = \sum_{i \geq 1} a_i x^i$ and $B(x) = \sum_{i \geq 1} b_i x^i$ such that all $a_i, b_i \geq 0$, and both $A(1) = \sum_i a_i = 6$ and $B(1) = \sum_i b_i = 6$ (since we want the new coins to be six-sided as well). Note that the sums start from $i = 1$, since we want all the numbers on both dice to be *positive*.
   It is not clear *a priori* that such a factoring exists. However, there does — to get this, let us first factor the GF for a single die.

   $$\begin{aligned} D(x) = x + x^2 + x^3 + x^4 + x^5 + x^6 &= x(1 + x + x^2 + x^3 + x^4 + x^5) \\ &= x(x^3 + 1)(x^2 + x + 1) \\ &= x(x + 1)(x^2 - x + 1)(x^2 + x + 1) \end{aligned}$$

   and hence the GF for the sum of the values of the pair of dice is

   $$P(x) = x^2(x + 1)^2(x^2 - x + 1)^2(x^2 + x + 1)^2$$

   Now if we rearrange these factors to form the two new polynomials

   $$A(x) = x(x + 1)(x^2 + x + 1) = x^4 + 2x^3 + 2x^2 + x \qquad (1)$$
   $$B(x) = x(x + 1)(x^2 - x + 1)(x^2 + x + 1)^2 = x^8 + x^6 + x^5 + x^4 + x^3 + x, \qquad (2)$$

   we note that these satisfy the criteria above, and in fact represent the two dice with sides $4, 3, 3, 2, 2, 1$ and $8, 6, 5, 4, 3, 1$ respectively.

Great Expectations

2. Which of the following statements are true in general:

   (a) For any two random variables $E[X + Y] = E[X] + E[Y]$.

      True. This is just linearity of expectations.

(b) For any two random variables $E[X \times Y] = E[X] \times E[Y]$.

False! Consider the experiment where you toss a coin, and let $X = 1$ if the toss comes up heads, and 0 otherwise. Let $Y = 1 - X$. Then $E[X] = 1/2 = E[Y]$, but $E[XY] = 0$.
Remember: $E[XY] = E[X]E[Y]$ only when X and Y are indepdendent of each other!

(c) For any random variable and a constant $c \in \mathbb{R}$, $E[c \cdot X] = c \cdot E[X]$.

True. Since the constant c is a random variable that is independent of X, it holds that $E[cX] = E[c]E[X] = c \cdot E[X]$.

(d) For any two *independent* random variables $E[X/Y] = E[X]/E[Y]$.

False! Let $X = 0$ with probability $1/2$ and 1 with probability $1/2$. Let $Y = 1$ with probability $1/2$ and 2 with probability $1/2$. Then $E[X] = 1/2$ and $E[Y] = 3/2$, giving $E[X]/E[Y] = 1/3$. But

$$E[X/Y] = 1/2 \cdot 0 + 1/4 \cdot 1/1 + 1/4 \cdot 1/2 = 1/4 + 1/8 = 3/8 \neq 1/3.$$

3. The Can-Do candy company sells chocolate truffles that have either cherry or raspberry centers. Alice really wants to eat a cherry truffle, so she decides to eat truffles until she gets a cherry truffle—she stops once she has eaten such a truffle. You may assume that each time she picks a truffle, it is equally likely to be cherry or raspberry. What is the expected number of cherry truffles Alice eats? What is the expected number of raspberry truffles she eats?

Since Alice stops eating cherry truffle once she ate one, she couldn't have possibly had more or less. So the expected number of cherry truffle she eats is 1. In order to count the expected number of raspberry truffle she eats, we can first calculate the expected number of cherry truffle she eats and subtract it by how many cherry truffle she eats.
We know that the expected number of truffle she eats is $\frac{1}{p}$ because this question is equivalent to flipping a coin with bias p and stopping once we get heads except instead of heads and tails, we have cherry and raspberry.
Therefore, expected number of truffles she eats is $\frac{1}{\frac{1}{2}} = 2$. Since one of them must have been a cherry truffle, the expected number of raspberry truffles she eats is also 1.

4. Bob is greedier than Alice, and will eat truffles until he has eaten 5 cherry truffles. What is the expected number of raspberry truffles he eats before he stops?

Bob's strategy is equivalent to repeating Alice's strategy five times. Since the expected number of raspberry truffles Alice eats before she stops is 1. Hence, if T_A was the random variable denoting the time at which Alice stopped, we know that $T_B = 5 \times T_A$. Moreover, from the previous part, we know that $E[T_A] = 1$. And by linearity of expectations, we have that $E[T_B] = E[5T_A] = E[T_A + T_A + T_A + T_A + T_A] = 5 \cdot E[T_A] = 5$. Hence tre expected number of raspberry truffles Bob eats before he stops is $5 \times 1 = 5$.

5. The TAs decide to distribute cookies to the students in class. There are $52 = 2 \times 26$ students in lecture; the TAs have $156 = 6 \times 26$ cookies. Each cookie is in the shape of a letter of the alphabet, and there are 6 A's, 6 B's, $\ldots 6$ Z's. Each student gets 3 cookies which he/she eats. If the cookies are distributed completely randomly to the students, what is the expected number of cookies that are eaten by students whose first name begins with the letter corresponding to the cookie?

For $i = 1 \ldots 52$ and $j = 1, 2, 3$, let X_{ij} represent the indicator random variable for the event "the i^{th} person's j^{th} cookie matches their name". I.e., $X_{ij} = 1$ when the j^{th} cookie eaten by ith person has the same letter as their first name, and 0 otherwise. The answer we are looking for is $E[\sum_{i=1}^{52}\sum_{j=1}^{3} X_{ij}]$, which is $\sum_{i=1}^{52}\sum_{j=1}^{3} E[X_{ij}]$ by linearity of expectations.

Now we know that $E[X_{ij}] = \frac{1}{26}$ because no matter what the person's first name is, the probability of getting a cookie with the same letter; this is because there are an equal number of cookies for each letter, and they are distributed uniformly at random. Hence the final answer is

$$\begin{aligned} E[\textstyle\sum_i \sum_j X_{ij}] &= 156 \times E[X_{ij}] \\ &= 156 \times \tfrac{1}{26} = 6. \end{aligned}$$

6. In the problem above, what is the expected number of students who get *at least* one cookie which matches their first name?

Let X_i represent the indicator random variable that is 1 if the person ate at least one cookie which matched their name, and 0 otherwise. $E[X_i] = 1 \times P(X_i = 1) = 1 - P(X_i = 0)$.

Now, let us calculate $P(X_i = 0)$. In order to *not* get a cookie which matches their name, imagine the process of the i^{th} child being given 3 cookies: she must get an incorrect first one with probability $1 - \frac{6}{156}$, an incorrect second one with probability $1 - \frac{6}{155}$, and an incorrect third one with probability $1 - \frac{6}{154}$. So

$$\begin{aligned} P(X_i = 0) &= (1 - \frac{6}{156})(1 - \frac{6}{155})(1 - \frac{6}{154}) \\ &= 0.8883 \end{aligned}$$

Which means that $E[X_i] = 1 \times (1 - P(X_i = 0)) = 0.1117$.

Finally, we want to know $E[\sum_i X_i]$, which by linearity of expectations, is $E[\sum_i X_i] = \sum_i E[X_i] = 26 \cdot 0.1117 = 2.9041$

7. You toss independently a fair coin and you cot the number of tosses until the first head appears. If this number is n, you receive 2^n dollars. What is the expected amount that you will receive?

Applying the definition of expected value, the expected payoff is $\sum_{i=1}^{\infty} \frac{1}{2^i} * (2^i) = \sum_{i=1}^{\infty} \frac{1}{2}$. The last summation diverges to infinity, so the expected payoff is unbounded.

Bayes' Rule

7. Suppose that with probability $\frac{2}{3}$, a randomly selected vehicle will pass inspection at a headlight inspection station. Assuming that successive vehicles pass or fail *independently* of one another, calculate the following probabilities.

(a) P(all of the next three vehicles inspected pass)

Let E_i be the event that ith vehicle that is inspected passes the inspection. We want to know $P(E_1 \cap E_2 \cap E_3)$. Since these three events are independent:

$$\begin{aligned} P(E_1 \cap E_2 \cap E_3) &= P(E_1)P(E_2)P(E_3) \\ &= \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{8}{27} \end{aligned}$$

(b) P(exactly one of the next three inspected passes)

Choose one car to be passed (this can be done in $\binom{3}{1}$ ways). The others must fail. Therefore, the probability of exactly one of the next three cars passing is

$$\binom{3}{1}\frac{2}{3}\frac{1}{3}\frac{1}{3} = \frac{2}{9}$$

(c) Given that at least one of the next three vehicles passes inspection, what is the probability that all three pass?

Based on the notation from the first part, we want to solve for:

$$P((E_1 \cap E_2 \cap E_3) \mid (E_1 \cup E_2 \cup E_3))$$

By the definition of conditional expectations, we have

$$\begin{aligned}
P((E_1 \cap E_2 \cap E_3) \mid (E_1 \cup E_2 \cup E_3)) &= \frac{P(E_1 \cap E_2 \cap E_3 \cap (E_1 \cup E_2 \cup E_3))}{P(E_1 \cup E_2 \cup E_3)} \\
&= \frac{P(E_1 \cap E_2 \cap E_3)}{P(E_1 \cup E_2 \cup E_3)} \\
&= \frac{\frac{8}{27}}{1 - P(E_1^c \cap E_2^c \cap E_3^c)} \\
&= \frac{\frac{8}{27}}{1 - \frac{1}{3}\frac{1}{3}\frac{1}{3}} \\
&= \frac{4}{13}
\end{aligned}$$

8. Suppose that an error in the dice manufacturing plant made it so that one out of every one hundered dice in production will always roll a six. If you pick a random die from the plant and roll it three times and each result is a six, what is the probability that the next roll will also be a six?

Let B be probability that the die is broken and S be the probability that the die rolls three sixes. $P(S) = P(B)P(S|B) + P(\overline{B})P(S|\overline{B}) = \frac{1}{100} * 1 + \frac{99}{100} * \frac{1}{216} = \frac{315}{100*216} = \frac{7}{480}$.
$P(B|S) = \frac{P(B \cap S)}{P(S)} = \frac{\frac{1}{100}}{\frac{7}{480}} = \frac{24}{35}$.
$P(\overline{B}|S) = \frac{P(\overline{B} \cap S)}{P(S)} = \frac{99480}{100*216*7} = \frac{11}{35}$.
So the probability of another six is $P(B) + P(\overline{B})\frac{1}{6} = \frac{24}{35} + \frac{11}{35}\frac{1}{6} = \frac{1}{35}\frac{24*6+11}{6} = \frac{31}{42}$