

15-251: Great Theoretical Ideas In Computer Science

Recitation 3 Solutions

Partitions

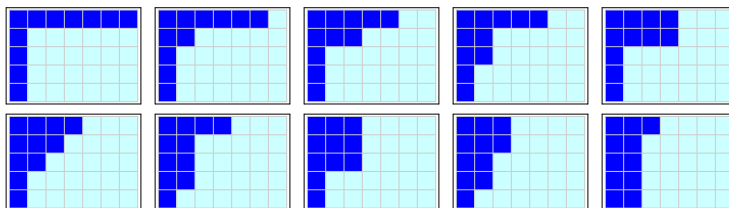
A partition of an integer is a decomposition of that integer into a sum of smaller integers. For example, there are five partitions of 4: (4), (3 + 1), (2 + 2), (2 + 1 + 1) and (1 + 1 + 1 + 1). The numbers in the sum are referred to as “terms.” Note that order does **not** matter in counting partitions. Two sums that differ only by the order in which the terms are written are the same. It is quite difficult to count the number of partitions for a given n , so we won’t get involved with that. Here are two slightly more reasonable questions.

1. Prove that the number of partitions of n with k parts is the same as the number of partitions of n whose largest term is k .

We need to construct a correspondence between the two different types of partitions. We will start by considering n divided into exactly k partitions. Consider such a partition: $n = a_1 + a_2 + \dots + a_k$. Order does not matter with partitions, so we imagine that the numbers are indexed in non-decreasing order: $a_1 \leq a_2 \leq \dots \leq a_{k-1} \leq a_k$.

Now we need to come up with a corresponding partition whose largest term is k . So, for $j \in \{1, 2, \dots, \max\{a_i\}\}$, let $b_j =$ the number of a_i such that $a_i \leq j$. Observe that each a_i is counted by b_1, b_2, \dots, b_{a_i} . In other words, each a_i is counted exactly a_i times. Thus $\sum_{j=1}^{\max\{a_i\}} b_j = \sum_{i=1}^k a_i = n$. Also, each $b_j \leq k$ because it includes at most all the a_i and $k = b_{\max\{a_i\}}$ because it does include all the a_i . Thus the largest term in the partition is exactly k . Thus the b_j form a partition of n whose greatest term is exactly k . Also, we can reverse this process by putting $a_i =$ the number of b_j such that $b_j \leq i$. Thus we satisfy the Sleuth’s Criteria.

This argument may seem technical but it is easier to understand visually. Consider the following example where $n = 11$ and $k = 5$. The rows of each diagram are the a_i and the



columns are the b_j .

2. Prove that the number of partitions of n with only odd terms is the same as the number of partitions of n with distinct terms.

Example: Letting $n = 6$, $\{(1 + 1 + 1 + 1 + 1 + 1), (3 + 1 + 1 + 1), (3 + 3), (5 + 1)\}$ is the set of partitions which have only odd terms; $\{(6), (5 + 1), (4 + 2), (3 + 2 + 1)\}$ is the set of partitions which have distinct terms.

Let D_n be the set of partitions of n with distinct terms, and let O_n be the set of partitions of n with odd terms. Note that any natural number can be written as the product of an odd number and a power of two; that is, for each n , $n = 2^k \cdot m$, where $k \geq 0$ and m is odd (this involves simply factoring out as many 2s out of n as possible; this can also be thought of as splitting up n into halves until this is no longer possible).

Let $f : D_n \rightarrow O_n$ be the function defined as follows: take a partition x from D_n and write each one of its terms as the product of an odd number and a power of two. Then, group the terms by their odd number. Finally, for each odd number, add up the corresponding

powers of two. $f(x)$ is the partition of n that contains each odd number repeated the resulting number of times. For example, the partition $(3 + 2 + 1)$ from D_6 is rewritten as $(2^0 \cdot 3 + 2^1 \cdot 1 + 2^0 \cdot 1)$, which is rewritten as $((2^0) \cdot 3 + (2^1 + 2^0) \cdot 1)$, which is $((1) \cdot 3 + (3) \cdot 1)$. The final result is the partition $(3 + 1 + 1 + 1)$.

Take two distinct partitions x, y from D_n and group their terms by the odd number, as described above. Because the terms of x are distinct from each other by the definition of D_n , and likewise for y , each group contains at most one of each power of two, so there is no way to obtain one power of two as a sum of others. Furthermore, it must be the case that for one of the odd numbers, m , the group in x is different from the group in y , because the partitions are different. Then, there is a power of two that is present in group m in x but absent in group m in y , or vice versa. The result is that the count of m terms in $f(x)$ is different from the count of m terms in $f(y)$, so $f(x) \neq f(y)$, and f is an injection.

Take any partition y from O_n , and count the number of instances of m in the partition; let that number be c_m . Then, rewrite the partition first as $(c_1 \cdot 1 + c_3 \cdot 3 + c_5 \cdot 5 + \dots + c_k \cdot k)$. Decompose each c_m into powers of two (that is, convert it to binary), and use the distributive law of multiplication to obtain another partition of n . Note that the powers of two resulting from decomposing one c_m are all distinct. Therefore, all the terms of the new partition are distinct, so the partition is in D_n , and f applied to this partition is exactly the original partition y . Therefore, f is surjective. f is a bijection, so $|D_n| = |O_n|$ as required.

Generating Functions

1. What is the power series for $\frac{1}{1-11x+30x^2}$?

$$\begin{aligned} \frac{1}{1-11x+30x^2} &= \frac{1}{(1-5x)(1-6x)} \\ &= \frac{6}{1-6x} - \frac{5}{1-5x} \\ &= 6 \sum_{i=0}^{\infty} 6^i x^i + 5 \sum_{i=0}^{\infty} 5^i x^i \\ &= \sum_{i=0}^{\infty} 6^{i+1} x^i + \sum_{i=0}^{\infty} 5^{i+1} x^i \\ &= \sum_{i=0}^{\infty} (6^{i+1} + 5^{i+1}) x^i \end{aligned}$$

2. Let $v_n = \langle 1, 2, 4, 8, 16, \dots \rangle$. Determine the generating function for this sequence.

The generating function for this sequence is $A(x) = \sum_{i \geq 0} 2^i \cdot x^i = 1x^0 + 2x^1 + 4x^2 + \dots$
 But this is just the infinite geometric sum $1 + (2x)^1 + (2x)^2 + (2x)^3 + \dots = \frac{1}{1-2x}$.

3. There are $p \geq 3$ pirates. Let c_n be the number of ways to distribute n pieces of gold to the pirates such that pirate 1 gets an even number of pieces, pirate 2 gets at least 5 pieces, and pirate 3 gets at most 3 pieces, and all other pirates can be given any number of gold pieces. Determine the generating function $C(z) = \sum_{i=0}^{\infty} c_i z^i$.

Solving this is like solving the fruit problem in class. Consider the number of valid ways to give n fruit to pirate #1: as we vary n , this is $1, 0, 1, 0, 1, 0, \dots$, since we can only give him n pieces when n is even. So if we let $P_1(x)$ be the generating function for this sequence, we get

$$P_1(x) = 1 + x^2 + x^4 + \dots = \frac{1}{1 - x^2}.$$

For pirate #2, the sequence is $0, 0, 0, 0, 0, 1, 1, 1, \dots$, since he must get at least 5 pieces: this gives the generating function

$$P_2(x) = x^5 + x^6 + \dots = x^5(1 + x + x^2 + \dots) = \frac{x^5}{1 - x}.$$

Pirate #3 gets at most 3 pieces, and hence his sequence is $1, 1, 1, 1, \dots$, which gives us

$$P_3(x) = 1 + x + x^2 + x^3 = \frac{1 - x^4}{1 - x}.$$

All the other $p - 3$ pirates can be given any number of pieces, so their sequences are $1, 1, 1, 1, 1, 1, \dots$, with generating functions (for $i > 3$):

$$P_i(x) = 1 + x + x^2 + x^3 + \dots = \frac{1}{1 - x}.$$

Now the generating function $P(x)$ for ways to divide up all the gold to the pirates is given by the product of these generating functions:

$$C(x) = \prod_{i=1}^p P_i(x) = \frac{1}{1 - x^2} \times \frac{x^5}{1 - x} \times \frac{1 - x^4}{1 - x} \times \left(\frac{1}{1 - x}\right)^{p-3}.$$

This can be simplified a little by noticing that $(1 - x^4) = (1 + x^2)(1 - x^2)$ to get

$$C(x) = \frac{x^5(1 + x^2)}{(1 - x)^{p-1}} = \frac{x^5}{(1 - x)^{p-1}} + \frac{x^7}{(1 - x)^{p-1}}.$$

We can now use this to find a closed form for c_n , the number of valid ways to distribute n pieces of gold among p pirates. The coefficient of x^n in the first term $\frac{x^5}{(1-x)^{p-1}}$ is the coefficient of x^{n-5} in $\frac{1}{(1-x)^{p-1}}$. But in class we saw that the coefficient of x^k in $\frac{1}{(1-x)^t}$ was $\binom{t+k-1}{t-1}$ — so the coefficient of x^{n-5} in $\frac{1}{(1-x)^{p-1}}$ is $\binom{p+n-7}{p-2}$.

Similarly, the coefficient of x^n in the second term $\frac{x^7}{(1-x)^{p-1}}$ is $\binom{p+n-9}{p-2}$. But now c_n , the coefficient of x^n in $C(x)$ is the sum of these two quantities: hence the number of ways to distribute n pieces of gold to these p pirates according to the rules given is

$$\binom{p+n-7}{p-2} + \binom{p+n-9}{p-2}.$$

4. Let a be the sequence $\langle a_0, a_1, a_2, \dots \rangle$, and b be the sequence $\langle b_0, b_1, b_2, \dots \rangle$. If $A(x)$ and $B(x)$ are the generating functions for a and b , find a generating function for the sequence $c = \langle a_0, b_0, a_1, b_1, a_2, b_2, \dots \rangle$.

The generating function for a is

$$A(x) = a_0 + a_1x + a_2x^2 + \dots = \sum_{i \geq 0} a_i x^i.$$

To get the generating function for $a_0, 0, a_1, 0, a_2, 0, \dots$, we note that this generating function will be

$$\begin{aligned}\widehat{A}(x) &= a_0 + a_1x^2 + a_2x^4 + \dots \\ &= a_0 + a_1(x^2)^1 + a_2(x^2)^2 + \dots = A(x^2).\end{aligned}$$

Similarly, if $B(x)$ is the generating function for b , then the generating function for $0, b_0, 0, b_1, 0, b_2, 0, \dots$ will be

$$\begin{aligned}\widehat{B}(x) &= b_0x + b_1x^3 + b_2x^5 + \dots \\ &= x(b_0 + b_1x^2 + b_2x^4 + \dots) \\ &= x \cdot B(x^2).\end{aligned}$$

Finally, note that the sequence for c is obtained by the component-wise addition of the two sequences $a_0, 0, a_1, 0, a_2, 0, \dots$ and $0, b_0, 0, b_1, 0, b_2, 0, \dots$. So the generating function for c is just the sum of the generating functions for these two sequences, which is $\widehat{A}(x) + \widehat{B}(x) = A(x^2) + x \cdot B(x^2)$.